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Abstract

We examine a noncooperative coalitional bargaining game model with renegotiations and provide a necessary and sufficient condition for the existence of a stationary subgame perfect equilibrium (SSPE) in which the grand coalition is immediately formed. When the discount factor is close to one, this condition indicates the existence of a Nash bargaining solution that is immune to any coalitional deviation, a condition that is equivalent to the nonemptiness of the core of the game. Additionally, we show that a strategic gradual coalition formation occurs when forming a sub-coalition gives a better inside option in the subsequent bargaining to the coalition members than to the outside players.

Keywords: Nash bargaining solution; Noncooperative bargaining; Coalitional deviation; Inside option.
JEL Classification: C72; C78.

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1 Introduction

This paper proposes a new kind of Nash bargaining solution (NBS) for games with externalities among coalitions. Concretely, we consider an NBS satisfying a certain stability property in which players will not collectively deviate from the NBS by forming a coalition to change inside options in a bargaining game. We call it a bargaining deviation-proof NBS and will present a noncooperative foundation for this NBS in the game.

A bargaining situation is described by a partition function form game (PFG), in which for every subset of the set of players and every partition of the set of players, a partition function assigns coalitional worth. Under some coalition structure π, each coalition is regarded as a representative player in bargaining. If they all cooperate to form a grand coalition, they can obtain maximum worth. Otherwise, each coalition obtains its worth under the given coalition structure. Its worth provides a disagreement payoff for each coalition. The NBS under this coalition structure is the profile of the payoffs for maximizing the product of the net payoffs for coalitions over their disagreement payoffs. Integrating some coalitions $S_1, \ldots, S_m$ in π to form a new coalition induces the formation of new coalition structure π'. Then, the NBS for coalitions in π' is derived because the disagreement point is altered. If the NBS payoff for the integrated coalition under π' is greater than the sum of the NBS payoffs for coalitions $S_1, \ldots, S_m$ in initial coalition structure π, these coalitions are said to have a *bargaining deviation* from the initial NBS under π. A bargaining deviation-proof NBS under π is one that is immune to coalitional deviations by any set of coalitions. The notion of a bargaining deviation-proof NBS corresponds to that of strong equilibrium by Aumann (1959) on the Nash equilibrium.

As our main result, we will show that the existence of a bargaining deviation-proof NBS for every coalition structure is a necessary and sufficient condition for an efficient stationary subgame perfect equilibrium (SSPE) of a noncooperative coalitional bargaining game to exist when players are sufficiently patient. We adopt a standard noncooperative bargaining game model with renegotiations as in Seidmann and Winter (1998), Okada (2000), Gomes (2005), Gomes and Jehiel (2005), Bloch and Gomes (2006), and Hyndman and Ray (2007). We focus on an efficient SSPE in which the grand coalition is immediately formed in any coalition structure.

Additionally, we define a new core concept for a PFG in which outside players obtain the NBS payoffs. The nonemptiness of the core is equivalent to the existence of a bargaining deviation-proof NBS in a bargaining game. In the core, there is no coalition that can be improved upon if the coalition expects outside players to obtain the NBS under the induced coalition structure. We call this core for a PFG the *Nash bargaining core* (NBC). The Nash core for a cooperative game in strategic form has been defined in the similar fashion in Okada (2010).

A considerable number of studies have been conducted on the relationship between the NBS, core, and and SSPE of a noncooperative bargaining game. Nash (1953) presented a noncooperative foundation for the NBS of a two-person game in his earlier work (Nash, 1950). This approach is called the Nash program. Chatterjee et al. (1993), Compte and Jehiel (2010), Miyakawa (2009), Okada (1996, 2010, 2011) and Yan (2002) considered the above relationship in the coalitional bargaining context. We should note that these studies considered a noncooperative coalitional bargaining game model in which renegotiations of coalitions are not allowed. On the other hand, although studies on a coalitional bargaining game with renegotiations are mainly concerned with convergence to the efficient grand coalition, little attention has been given to the relationships between the NBS, core, and efficient SSPE of a noncooperative bargaining game. Seidmann and Winter (1998) showed that there is no efficient SSPE of a bargaining game without externalities if the core of the corresponding cooperative game is empty. Okada (2000) provided a necessary and sufficient condition for the existence of an efficient SSPE of a bargaining game without externalities. Gomes (2005) presented a *sufficient* condition that guarantees the existence of an efficient SSPE: the grand coalition is formed immediately in a bargaining game with externalities. We fully characterize the efficient SSPE of a bargaining game with externalities through giving a *necessary and sufficient* condition for it to exist, and clarify the relationships between the NBS and the core of the game.

We will give another interpretation of a noncooperative coalitional bargaining game with renegotiations. In the coalitional bargaining game model with externalities but without renegotiations, such as Bloch (1996), Ray and Vohra (1999), Okada (2010), and Compte and Jehiel (2010), coalition members would leave the game if they formed a coalition, and the rest of the players continued negotiating. Forming coalitions implies a commitment by coalition members to withdraw from negotiations. Thus, forming a sub-coalition provides "outside options" in bargaining. The relationships between the NBS and efficient SSPE of the coalitional bargaining game model without renegotiations are examined in our companion paper Kawamori and Miyakawa (2011). On the other hand, in the coalitional bargaining game model with renegotiations, such as Goems (2005), Gomes and Jehiel (2005), and Bloch and Gomes (2006), the members of coalitions stay in the bargaining game and receive their payoffs for every bargaining round. The worth of coalitions in a given coalition structure is regarded as a disagreement payoff for each coalition. Thus, forming a sub-coalition provides "inside options" in bargaining in this setting. Bloch and Gomes (2006) considered outside options by coalitions in the coalitional bargaining game model with inside options.

Using the interpretation of coalition formations as inside options, we reconsider a symmetric three-player bargaining game. We will observe that if a necessary and sufficient condition for the existence of efficient SSPE of the bargaining game is not satisfied, an SSPE will exist in which the grand coalition is gradually formed (i.e., in two steps). In such a case, the proposer in the first stage gains an advantage over the outside player in the succeeding bargaining stage by forming a two-player coalition because the inside option for the representative of the two-player coalition is relatively better than that for the outside player. Through the effect of inside options, a gradual coalition formation emerges strategically. Seidmann and Winter (1998) provided some examples of gradual coalition formation and an immediate move towards grand coalition in a bargaining game without externalities (Theorem 1 and Proposition 1 in Seidmann and Winter, 1998). Our result is an extension of the Seidmann and Winter model's results. Their results are explained as being a special case of our result from the viewpoint of inside options in the coalitional bargaining game.

The remainder of the paper is organized as follows. Section 2 describes the bargaining situation and provides a noncooperative coalitional bargaining game model. Section 3 defines the bargaining deviation-proof NBS and the Nash bargaining core. Section 4 states the main results. Section 5 considers a gradual coalition formation in a three-player case and provides two applications: a public goods economy and horizontal mergers in a Cournot market. Section 6 concludes. Proofs of all propositions are provided in the

Appendix.

2 Model

We consider a multi-person bargaining model as follows.

For any set S, let $\bar{S} := \bigcup S$. For any sets S and T, let $S|T := S \setminus T \cup \{\bar{T}\}$. For any sets X and Y, let Y^X be the set of functions from X to Y. For any function f and any element x of the domain of f, let f_x be the image of x under f, i.e., $f_x := f(x)$. For any nonempty set S, let the set of partitions of S be denoted by Π_S.

The underlying bargaining situation is represented by PFG (N, v); that is, a pair (N, v) such that N is a nonempty finite set and v is a function from $\mathcal{C} := \{(S, \pi) \in 2^N \times \Pi_N | S \in \pi\}$ to $\mathbb{R}_+$. $i \in N$ and $S \in 2^N \setminus \{\emptyset\}$ are called a *player* and a *coalition*, respectively. $v(S, \pi)$ represents the worth of coalition S under coalition structure π. For any $(S, \pi) \in \mathcal{C}$, let v_S^π denote $v(S, \pi)$. We assume that the grand coalition is efficient in (N, v), i.e., for any $\pi \in \Pi_N$,

$$v_N^{\{N\}} > \sum_{S \in \pi} v_S^\pi.$$

For $\pi \in \Pi_N$, let $[\cdot]_\pi : N \to \pi$ be the projection of the equivalence relation induced by π. For any $(\pi, S) \in \Pi_N \times 2^N$, let $[\cdot]_{\pi S}$ be the restriction of $[\cdot]_\pi$ to S.

We define an extensive form game as follows. A *state* of the game is $(\pi, M) \in \Pi_N \times 2^N$ such that M is a complete system of representatives for π, i.e., $[\cdot]_{\pi M}$ is bijective. M is the set of active players in this extensive form game, and π is a coalition structure. Player $i \in M$ possesses the decision rights of players in $[i]_\pi$, i.e, the decision right of players in coalition $I \in \pi$ is possessed by player $[I]_{\pi M}^{-1}$. $i \in M$ possesses the decision rights of players in $[i]_\pi$. In a round with state (π, M), the bargaining proceeds as follows.

(i) A coalition I in π is selected with probability $1/|\pi|$, and player $[I]_{\pi M}^{-1}$ becomes the proposer in this round.

(ii) The proposer proposes an element of

$$X_I^\pi := \left\{ (\rho, x) \middle| \rho \in 2^\pi \wedge I \in \rho \wedge x \in \mathbb{R}^\rho \wedge \sum_{J \in \rho} x_J = 0 \right\}.$$

Proposal $(\rho, x) \in X_I^\pi$ means that the proposer chooses coalition $[\rho]_{\pi M}^{-1}$ and offers x_J to player $[J]_{\pi M}^{-1}$ for any $J \in \rho$.

(iii) Each player in $[\rho]_{\pi M}^{-1}$ accepts or rejects the proposal according to some predetermined order.

(iv) The state transits to (π', M') as follows.

(a) If all players accept the proposal, for any $J \in \rho \setminus \{I\}$, player $[J]_{\pi M}^{-1}$ gets x_J, transfers his decision rights to player $[I]_{\pi M}^{-1}$, and leaves the game. Moreover, player $[I]_{\pi M}^{-1}$ obtains x_I and remains in the game as a representative of coalition $\bar{\rho}$. The state transits to $(\pi', M') = \left(\pi|\rho, M \setminus [\rho]_{\pi M}^{-1} \cup \{[I]_{\pi M}^{-1}\}\right)$.

(b) If some player rejects the proposal, the state is unchanged, that is, $(\pi', M') = (\pi, M)$.

For any $J \in \pi'$, player $[J]_{\pi M}^{-1}$ obtains her payoff $(1 - \delta)\, v_J^{\pi'}$ per period, where $\delta \in [0, 1)$ is the discount factor. After this, the game goes to the next round.

Future payoffs are geometrically discounted by δ. We denote the extensive form game under the discount factor δ by $G(\delta)$.

Remark 1. If the probability distribution for the proposer to be selected among active players is generalized to $p^\pi \in \Delta(\pi)$ at coalition structure π, our extensive form game corresponds to the coalitional bargaining game in Gomes (2005). The following argument can be applied to the generalized model. A one-to-one correspondence between the weight of the product of net payoffs among players in the NBS and the probability distribution under which the proposer is selected in a noncooperative bargaining game with random proposers was shown in Miyakawa (2008), Britz, Herrings, and Predtetchinski (2010), and Okada (2010, 2011). We will assume that $p_I^\pi = 1/|\pi|$ for all $I \in \pi$ to define the bargaining deviation-proof NBS and the Nash bargaining core simply. The relationships to the Gomes (2005) model will be examined in Section 4.3.

3 Nash bargaining solution and core

3.1 Nash bargaining solution

Let us define the Nash bargaining solution under partition π. If an agreement is not reached, each coalition in π stands alone, and thus, coalition $I \in \pi$ receives v_I^π. Therefore, $(v_I^\pi)_{I\in\pi}$ presents a disagreement point under π. The set of feasible payoff allocations through possible agreements among coalitions in π is given by

$$B^\pi := \left\{ x \in \mathbb{R}^\pi \middle| \sum_{I\in\pi} x_I \le v_N^{\{N\}} \right\}.$$

The bargaining problem under π is a pair $\left(B^\pi, (v_I^\pi)_{I\in\pi}\right)$. We regard an element and a nonempty subset of π as a player and a coalition, respectively.

Definition 1. For any $\pi \in \Pi_N$, the *Nash bargaining solution (NBS)* $b^\pi = (b_I^\pi)_{I\in\pi}$ under π is a Nash bargaining solution of the bargaining problem $\left(B^\pi, (v_I^\pi)_{I\in\pi}\right)$, that is, a solution of the maximization problem $\max_{x\in B^\pi} \prod_{I\in\pi} (x_I - v_I^\pi)$.

Under the assumption of transferable utilities, for any $\pi \in \Pi_N$ and any $I \in \pi$, the Nash bargaining solution under π is given by

$$b_I^\pi = \frac{v_N^{\{N\}} - \sum_{J\in\pi} v_J^\pi}{|\pi|} + v_I^\pi.$$

Definition 2. For any $\pi \in \Pi_N$, any $\rho \in 2^\pi \setminus \{\emptyset\}$ and any $x \in B^\pi$, ρ has a *bargaining deviation* from allocation x under π if $b_{\bar\rho}^{\pi|\rho} > \sum_{I\in\rho} x_I$. For any $\pi \in \Pi_N$, a *bargaining deviation-proof* allocation under π is $x \in B^\pi$ such that if for any $\rho \in 2^\pi \setminus \{\emptyset\}$, ρ does not have a bargaining deviation from x under π.

3.2 Nash bargaining core

Using an NBS under π, we define a characteristic function form game as follows. Let V^π be the function from $2^\pi \setminus \{\emptyset\}$ to $\mathbb{R}$ such that for any $\rho \in 2^\pi \setminus \{\emptyset\}$,

$$V_\rho^\pi = b_{\bar\rho}^{\pi|\rho}.$$

Hence, (π, V^{π}) is a characteristic function form game. We regard an element and a nonempty subset of π as a player and a coalition, respectively. The features of this definition are as follows: When a coalition ρ is formed in π and coalition structure $\pi|\rho$ is induced, the members of ρ expect that ρ will obtain its share in the NBS under the induced coalition structure and this share will be the worth of ρ in π. We give an alternative interpretation of the characteristic function form game (π, V^{π}). The maximum worth for all coalitions is given by $v_N^{\{N\}}$. The above characteristic function is also represented by $V_\rho^\pi = v_N^{\{N\}} - \sum_{I\in\pi\setminus\rho} b_I^{\pi|\rho}$. Thus, if the members of $\rho \in 2^\pi \setminus \{\emptyset\}$ form a coalition $\bar{\rho}$, outside coalitions $I \in \pi \setminus \rho$ obtain NBS $b_I^{\pi|\rho}$ and $\bar{\rho}$ obtains the remainder of $v_N^{\{N\}}$.

Definition 3. For any $\pi \in \Pi_N$, the *Nash bargaining core* (NBC) C^π under π is the core of the characteristic function form game (π, V^π), i.e.,

$$\left\{x \in \mathbb{R}^\pi | \sum_{I\in\pi} x_I \leq V_\pi^\pi \wedge \forall \rho \in 2^\pi \setminus \{\emptyset\}, \sum_{I\in\rho} x_I \geq V_\rho^\pi \right\}.$$

Remark 2. Our core concept is closely related to the Nash core in Okada (2010). He considered a cooperative game in strategic form $\left(N, (A_i)_{i\in N}, (u_i)_{i\in N}\right)$, where N is the set of players, A_i is the set of actions and u_i is a payoff function for player i. In this bargaining game, players choose not only a coalition but also a correlated action of the coalition. On the other hand, we start with the partition function form game (N, v). For each coalition structure π, a correlated action $p_S^\pi \in \Delta(A_S)$ of every coalition is predetermined in the partition function form game. In Okada (2010), a characteristic function V is defined such that the members of complementary coalition $N \setminus S$ choose the NBS. Furthermore, the NBS and the disagreement point for $N \setminus S$ depend on correlated action p_S of coalition S. Concretely, the characteristic function is defined for all $S \subset N$ as

$$V_S = \left\{x \in \mathbb{R}^S | \exists p_S \in \Delta(A_S), \forall i \in S, u_i(p_S, b^*(p_S)) \geq x_i \right\},$$

where u_i is the expected payoff function for player i and $b^*(p_S)$ is the correlated actions to realize the NBS for the members of the complementary coalition $N \setminus S$. However, in this paper, there is no interdependence between the correlated actions of coalitions and the NBS for outside players or coalitions because the correlated actions are predetermined in the partition function form game.

4 Efficient equilibrium

4.1 Characterization of efficient SSPE

Our equilibrium concept in the noncooperative games is a stationary subgame perfect equilibrium (SSPE). An SSPE satisfies the subgame perfectness and the stationarity property in that for any states (π, M) and (π, M'), and for any $I \in \pi$, (i) player $[I]^{-1}_{\pi M}$'s proposal in any round with state (π, M) is the same as player $[I]^{-1}_{\pi M'}$'s proposal in any round with state (π, M') and (ii) for any proposal (ρ, x) to be responded to by player $[I]^{-1}_{\pi M}$ in any round with state (π, M) and for any proposal (ρ, x') to be responded to by player $[I]^{-1}_{\pi M'}$ in any round with state (π, M') such that $x_I = x'_I$, player $[I]^{-1}_{\pi M}$'s response to (ρ, x) in any round with state (π, M) is the same as player $[I]^{-1}_{\pi M'}$'s response to (ρ', x') in any round with state (π, M').

Definition 4. For any $\delta \in [0, 1)$, a strategy profile σ of $G(\delta)$ is *efficient* if in any round with state (π, M), every player proposes partition π when she is a proposer and accepts proposals in σ.

For any $\delta \in [0, 1)$ and any strategy profile σ of $G(\delta)$ that satisfies the stationarity property, $u : \mathcal{C} \to \mathbb{R}$ is called the *payoff configuration* of σ under δ if for any π and any $I \in \pi$, for some complete system M of representatives of π, u^π_I is the expected payoff of player $[I]^{-1}_{\pi M}$ by σ in the subgame starting from (π, M). By the stationarity, for any $\delta \in [0, 1)$ and any grand-coalition efficient SSPE σ, there uniquely exists a payoff configuration of σ under δ.

Proposition 1 characterizes efficient SSPEs.

Proposition 1. *Let $\delta \in [0, 1)$. Let σ be an efficient SSPE of $G(\delta)$. In any round with state (π, M), for any $I \in \pi$, player $[I]^{-1}_{\pi M}$'s proposal in σ is (π, x) such that*

$$x_I = (\delta + (1 - \delta)|\pi|) \frac{v^{\{N\}}_N - \sum_{K \in \pi} v^\pi_K}{|\pi|} + v^\pi_I$$

and for any $J \in \pi \setminus \{I\}$,

$$x_J = \delta \frac{v_N^{\{N\}} - \sum_{K \in \pi} v_K^{\pi}}{|\pi|} + v_J^{\pi},$$

and the proposal is accepted in σ*. For any state* (π, M) *and any* $I \in \pi$*, player* $[I]_{\pi M}^{-1}$*'s expected payoff by* σ *is* b_I^{π}*; that is, payoff configuration* u *of* σ *under* δ *is such that for any* $(I, \pi) \in \mathcal{C}$*,* $u_I^{\pi} = b_I^{\pi}$*.*

From Proposition 1, in any round with state (π, M), as the discount factor tends to unity, the allocation in any player's proposal in an efficient SSPE converges to the NBS under π. This implies that in state (π, M), each player's proposal in an efficient SSPE converges to the same payoff allocation, which is the NBS under π.

4.2 Condition for efficient agreements

We provide a necessary and sufficient condition for an efficient SSPE of $G(\delta)$ to exist.

Proposition 2. *Let* $\delta \in [0, 1)$*. There exists an efficient SSPE of* $G(\delta)$ *if and only if for any* $\pi \in \Pi_N$ *and any* $\rho \in 2^{\pi} \setminus \{\emptyset\}$*,*

$$((1-\delta)|\pi| + \delta|\rho|) \frac{v_N^{\{N\}} - \sum_{J \in \pi} v_J^{\pi}}{|\pi|} + \sum_{I \in \rho} v_I^{\pi} \geq \delta \frac{v_N^{\{N\}} - \sum_{J \in \pi|\rho} v_J^{\pi|\rho}}{|\pi| - |\rho| + 1} + v_{\bar{\rho}}^{\pi|\rho}. \tag{1}$$

Next, consider a situation where the discount factor is close to one. From Proposition 2, we have the following proposition.

Proposition 3. *The following statements (i), (ii) and (iii) are equivalent:*

(i) For some $\bar{\delta} \in [0, 1)$*, for any* $\delta \in [\bar{\delta}, 1)$*, there exists an efficient SSPE of* $G(\delta)$*.*

(ii) For any $\pi \in \Pi_N$*, the NBS under* π *is a bargaining deviation-proof allocation.*

(iii) For any $\pi \in \Pi_N$*, the NBC under* π *is nonempty.*

4.3 The generalized model: Relationship with Okada (2000) and Gomes (2005)

We have two preceding studies to provide a condition for the existence of SSPE in which all players agree to form the grand coalition immediately in a coalitional bargaining game model with renegotiations. Okada (2000) presents a necessary and sufficient condition for the existence of a grand coalition efficient SSPE in the characteristic function form game. Gomes (2005) provides a sufficient condition for the existence of a grand coalition efficient SSPE in the partition function form game. By generalizing a probability distribution for the proposer selection, we can clarify the relationship of our necessary and sufficient condition (1) in Proposition 2 with the ones in Okada (2000) and Gomes (2005).

More generally, let us assume that one of the active players $[I]^{-1}_{\pi M}$ is randomly chosen with probability p^{π}_I under π in the coalitional bargaining game model. We denote this bargaining game model by $G^*(\delta)$; such a model was considered in Gomes (2005). Our model in the previous section is a case in which $p^{\pi}_I = 1/|\pi|$ for any $I \in \pi$. Okada (2000) considered a case in which $p^{\pi}_I = |I|/|N|$ for any $I \in \pi$. By employing the same argument for the game $G^*(\delta)$, we have the following proposition.

Proposition 4. *Let $\delta \in [0,1)$. There exists an efficient SSPE of $G^*(\delta)$ if and only if for any $\pi \in \Pi_N$ and for any $\rho \in 2^{\pi} \setminus \{\emptyset\}$,*

$$\left((1-\delta)+\delta\sum_{I\in\rho}p^{\pi}_I\right)\left(v^{\{N\}}_N-\sum_{I\in\pi}v^{\pi}_I\right)+\sum_{I\in\rho}v^{\pi}_I \geq \delta p^{\pi|\rho}_{\bar{\rho}}\left(v^{\{N\}}_N-\sum_{I\in\pi|\rho}v^{\pi|\rho}_I\right)+v^{\pi|\rho}_{\bar{\rho}}. \quad (2)$$

For any efficient SSPE σ of $G^*(\delta)$, the payoff configuration u of σ is such that for any $(I,\pi)\in\mathcal{C}$, $u^{\pi}_I = p^{\pi}_I\left(v^{\{N\}}_N-\sum_{J\in\pi}v^{\pi}_J\right)+v^{\pi}_I$.

We obtain the proposition by repeating the same argument; thus, we omit the proof.

Gomes (2005)'s condition corresponds to (2) at $\delta = 1$. Gomes's condition implies condition (2) because the left hand side and right hand side of (2) is increasing and decreasing in δ, respectively. Therefore, his condition is sufficient for an efficient SSPE of $G^*(\delta)$ to exist.

Okada (2000) considered the 0-normalized characteristic function form game (N,V). Thus, $V_S = v^{\pi}_S$ for any $(S,\pi)\in\mathcal{C}$ and $V_{\{i\}} = 0$ for any $i \in N$. Condition (2) at state

$(\{\{i\} \,| i \in N\}, N)$ is reduced to

$$\left((1-\delta)+\delta\frac{|S|}{|N|}\right)V_N \geq \delta\frac{|S|}{|N|}\left(V_N - V_S\right) + V_S.$$

This inequality is rewritten as

$$(1-\delta)\,V_N \geq \left(1-\delta\frac{|S|}{|N|}\right)V_S. \tag{3}$$

This is condition (10) in Okada (2000), which is a necessary and sufficient condition for an efficient SSPE in a characteristic form game to exist.

5 Strategic coalition formation

5.1 Gradual coalition formation: three-player case

Let us consider a three-symmetric-player case: $|N| = 3$. We denote the coalition structure with the grand coalition $\{N\}$ by $\overline{\pi}$, the coalition structure with two-player coalition $\{I, N \setminus I\}$ by π_I for any $I \in 2^N$ with $|I| = 1$ and the coalition structure with singleton coalitions $\{\{i\} \,| i \in N\}$ by $\underline{\pi}$. We simply describe the partition function for any $I \in \underline{\pi}$ as

$$v_N^{\overline{\pi}} = \overline{v}, \quad v_{N\setminus I}^{\pi_I} = v_2, \quad v_I^{\pi_I} = v_1, \quad v_I^{\underline{\pi}} = \underline{v}.$$

All players are symmetric; hence, for any $\delta \in [0,1)$, a necessary and sufficient condition (1) for the existence of an efficient SSPE of $G(\delta)$ is presented by

$$\delta \leq \frac{6\left(\overline{v} - v_2 - \underline{v}\right)}{3\left(\overline{v} - v_2 - v_1\right) + 2\left(\overline{v} - 3\underline{v}\right)}. \tag{4}$$

This condition implies that the grand coalition is formed immediately if and only if the discount factor is below a critical value. Moreover, if $\overline{v} - 3v_2 + 3v_1 \geq 0$, there exists an efficient SSPE for any $\delta \in [0,1)$; if $\overline{v} - v_2 - \underline{v} < 0$, no efficient SSPE of $G(\delta)$ exists for any $\delta \in [0,1)$; otherwise, the right hand side of (4) is in $[0,1)$. Furthermore, as $\delta \to 1$,

(4) is reduced to

$$v_2 - \frac{1}{3}\overline{v} \leq v_1. \tag{5}$$

Therefore, if (5) is not satisfied and $\delta \to 1$, there is an SSPE of $G(\delta)$ in which coalitions are formed gradually: a two-player coalition is formed firstly, and then, the grand coalition is formed. We say that a strategy profile σ is *gradual-coalition-formation* strategy profile if in σ, in any round with state $(\underline{\pi}, N)$, any player proposes a two-player coalition. We say that a strategy profile σ is a *symmetric stationary subgame perfect equilibrium (SSSPE)* if σ is an SSPE and for some generator g such that $\underline{\pi}$ is a cyclic group with g, for any $I \in \underline{\pi}$, $\rho_{Ig} = \{Jg | J \in \rho_I\}$, where for any $I \in \underline{\pi}$, ρ_I is the coalition that player $[I]^{-1}_{\underline{\pi}N}$ proposes in state $(\underline{\pi}, N)$ in σ. Note that for any SSSPE σ, σ is an efficient or gradual-coalition-formation SSSPE, because any proposal of a singleton coalition is equivalent to the delay, but any SSPE involves no delay by Lemma 2 in the Appendix.

For any $\delta \in [0, 1)$, let u^δ be the function from $\mathcal{C}$ to $\mathbb{R}$ such that for any $I \in \underline{\pi}$,

$$\begin{aligned}
u_I^{\delta\underline{\pi}} &= \frac{1}{3}\left(\delta\overline{v} + (1-\delta)\, v_1 + (1-\delta)\, v_2\right), \\
u_I^{\delta\pi_I} &= \frac{1}{2}\left(\overline{v} + v_1 - v_2\right), \quad u_{N\setminus I}^{\delta\pi_I} = \frac{1}{2}\left(\overline{v} - v_1 + v_2\right), \\
u_N^{\delta\overline{\pi}} &= \overline{v}.
\end{aligned} \tag{6}$$

Note that for any $I \in \underline{\pi}$, $\lim_{\delta \to 1} u_I^{\delta\underline{\pi}} = (1/3)\,\overline{v}$.

Proposition 5 characterizes the payoff configuration of any gradual-coalition-formation SSSPE.

Proposition 5. *Let $\delta \in [0, 1)$. Let σ be a gradual-coalition-formation SSSPE of $G(\delta)$. Then, the payoff configuration of σ is u^δ.*

Proposition 6 provides a necessary and sufficient condition for a gradual-coalition-formation SSSPE to exist.

Proposition 6. *Let $\delta \in [0, 1)$. There exists a gradual-coalition-formation SSSPE of $G(\delta)$*

if and only if

$$\frac{2}{3}\left(\overline{v}-v_1-v_2\right)\delta^2+\frac{1}{3}\left(\left(\overline{v}-v_1-v_2\right)+2\left(\overline{v}-3\underline{v}\right)\right)\delta-2\left(\overline{v}-\underline{v}-v_2\right)\geq 0. \tag{7}$$

Remark 3. Let L be the function from $[0,1)$ to $\mathbb{R}$ such that for any $\delta\in[0,1)$, $L(\delta)$ is equal to the left hand side of (7). L is continuous and strictly increasing. Thus, condition (7) implies that the grand coalition is gradually formed if and only if the discount factor is above a critical value. Moreover, if $L(0)\geq 0$, i.e., $\overline{v}-\underline{v}-v_2\leq 0$, there exists a gradual-coalition-formation SSSPE of $G(\delta)$ for any $\delta\in[0,1)$; if $L(1)\leq 0$, i.e., $\overline{v}+3v_1-3v_2\geq 0$, there does not exist a gradual-coalition-formation SSSPE of $G(\delta)$ for any $\delta\in[0,1)$; otherwise, for some $\bar{\delta}\in(0,1)$, for any $\delta\in[0,1)$, there exists a gradual-coalition-formation SSSPE of $G(\delta)$ if and only if $\delta\geq\bar{\delta}$.

By Proposition 6 and (4), we have Proposition 7, which characterizes SSSPE when the discount factor is large.

Proposition 7. *(i) If $v_2-(1/3)\,\overline{v}>v_1$, then, for some $\bar{\delta}\in[0,1)$, for any $\delta\in\left[\bar{\delta},1\right)$, there exists a gradual-coalition-formation SSSPE of $G(\delta)$, while no efficient SSSPE of $G(\delta)$ exists. (ii) If $v_2-(1/3)\,\overline{v}\leq v_1$, then, for some $\bar{\delta}\in[0,1)$, for any $\delta\in[\bar{\delta},1)$, there exists an efficient SSSPE of $G(\delta)$, while no gradual-coalition-formation SSSPE of $G(\delta)$ exists.*

Figure 1 summarizes our results of the immediate move to the grand coalition and the gradual coalition formation.

The point $(v_1, v_2-(1/3)\,\overline{v})$ represents the inside option in the bargaining game in the next round after the proposer selects a two-player coalition. Forming a sub-coalition is regarded as the selection of the inside option in the subsequent bargaining. The inside options are endogenously determined through the formation of coalitions in the bargaining game model. The final limit payoff profile for players 1 and 3 when player 1 chooses the two-player coalition $\{1,2\}$ is represented by an intersection between a straight line through $(v_1, v_2-(1/3)\,\overline{v})$ paralleled to 45-degree line and the frontier of the feasible payoff allocation for players 1 and 3 as $\delta\to 1$. If the final payoff for player 1 is greater than $(1/3)\,\overline{v}$, then, player 1 selects the two-player coalition $\{1,2\}$ in SSSPE. That is, if

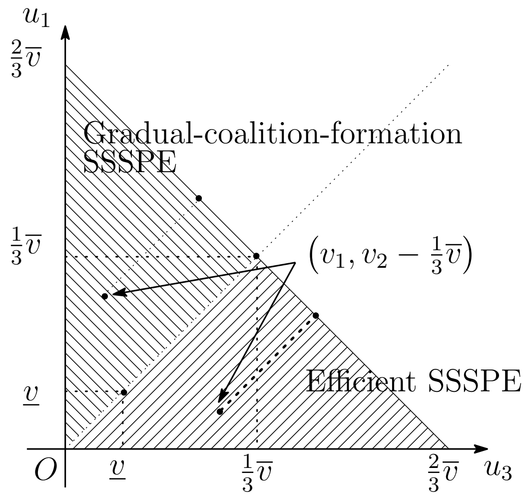

Figure 1: Effect of inside options

$(v_1, v_2 - (1/3)\,\overline{v})$ is in the region above the 45-degree line, then, there exists a gradual-coalition-formation SSSPE because the proposer selects a two-player coalition when δ is close to one. On the other hand, when $(v_1, v_2 - (1/3)\,\overline{v})$ belongs to the region below the 45-degree line, there exists an efficient SSSPE as $\delta \to 1$.

Seidmann and Winter (1998) considered a gradual coalition formation in the bargaining game that is based on a characteristic function form game. Let us reexamine their argument in the three symmetric player case as $\delta \to 1$. They considered a characteristic function form game (N, V) such that $N = \{1, 2, 3\}$ and for some $v_2 \in [0, 1]$, for any $S \in 2^N \setminus \{\emptyset\}$,

$$V_S = \begin{cases} 1 & \text{if } |S| = 3 \\ v_2 & \text{if } |S| = 2 \\ 0 & \text{if } |S| = 1. \end{cases}$$

Since the characteristic function form game is a transferable utility game, a necessary and sufficient condition for the core of (N, V) to be nonempty is that for any $S \in 2^N \setminus \{\emptyset\}$, $V_N / |N| \geq V_S / |S|$, i.e.,

$$\frac{1}{3} \geq v_2 - \frac{1}{3}. \tag{8}$$

Suppose that player 1 is selected as a proposer. Applying our argument to their setting, the inside option by forming a two-player coalition $\{1, 2\}$ is given by point $(0, v_2 - 1/3)$, which is on the vertical line through the initial state $(0, 0)$.

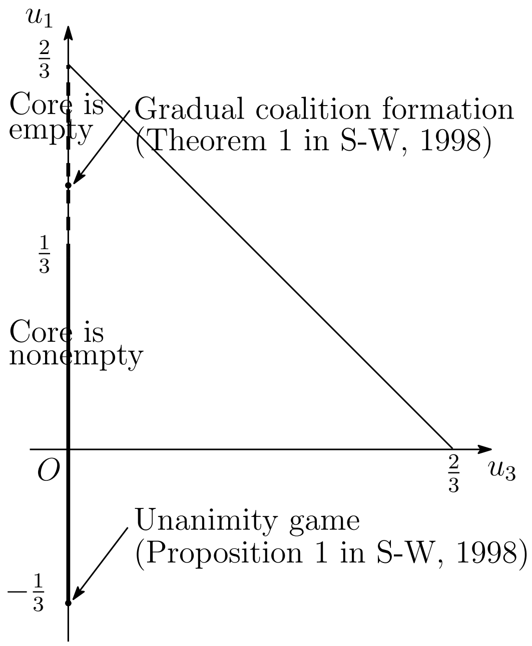

Figure 2: Seidmann and Winter (1998)

By condition (8), if the inside option $(0, v_2 - 1/3)$ in the bargaining game belongs to the segment of the vertical axis such that $1/3 < u_1 \leq 2/3$, the core of the game is empty. If the inside option belongs to the segment such that $-1/3 \leq u_1 \leq 1/3$, the core of the game is nonempty.

Theorem 1 in Seidmann and Winter (1998) says that if the core of the game is empty, then, there is no SSPE in which the grand coalition is formed immediately as $\delta \to 1$. Thus, if the inside option in the subsequent bargaining game is in the upper segment of the vertical line, player 1 obtains at least 1/3, regardless of the distribution of bargaining power in the bilateral bargaining between players 1 and 3. Proposition 1 in Seidmann and Winter (1998) claims that if the game is a unanimity game, that is, $v_2 = 0$, there exists an efficient SSPE as $\delta \to 1$. In Figure 2, the inside option for players 1 and 3 is on the point $(0, -1/3)$ when player 1 forms two-player coalition $\{1, 2\}$. Even if she has all the bargaining power in the subsequent bilateral bargaining, player 1 obtains at the most 1/3. Then, player 1 forms the grand coalition immediately in equilibrium.

5.2 Applications

Public goods economy Consider a public goods economy as in Ray and Vohra (2001). There are three individuals 1, 2, and 3. They can produce a public good. Contribution to the public good by a player is non-rivalrously enjoyed by all players, and it costs a player $c(z)$ to produce z units of public good. If Z is the total amount of the public good, the payoff for a player who produces z is given by $Z - c(z)$. Under coalition structure π, each coalition $S \in \pi$ decides a contribution profile of coalition members to maximize the sum of coalition members' payoffs, given the contributions of players outside of S. This situation is a strategic form game, where the set of players is π. Assume that $c(z) = (1/2) z^2$. Then, in the Nash equilibrium under π, by additively separability of payoff functions, player S decides member i's contribution as it maximizes $|S| z_i - c(z_i)$, and thus, the contribution of any member in $S \in \pi$ is $|S|$. Assume that monetary transfers are possible within each coalition. Then, player S's payoff of the unique Nash equilibrium of the strategic form game under π is the worth of S under π. Therefore, $\overline{v} = 3(3 \cdot 3 - (1/2) 3^2) = 27/2$, $v_2 = 2((2 \cdot 2 + 1) - (1/2) 2^2) = 6$, $v_1 = (2 \cdot 2 + 1) - (1/2) 1^2 = 9/2$ and $\underline{v} = 3 \cdot 1 - (1/2) 1^2 = 5/2$.

Under this situation, $\overline{v} + 3v_1 - 3v_2 = 27/2 + 3(9/2) - 3 \cdot 6 = 9 \geq 0$. Therefore, there exists an efficient SSSPE of $G(\delta)$, while no gradual-coalition-formation one exists for any $\delta \in [0, 1)$.

R&D alliances Bloch (1995) considered mergers with synergies in a Cournot oligopoly market. There are three firms 1, 2 and 3. Inverse demand function $P : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $P(Q) = \mathbf{1}_{\alpha - Q > 0}(\alpha - Q)$ for some $\alpha \in \mathbb{R}_{++}$. Forming a coalition for R&D reduces the marginal cost of production. We assume that a constant marginal cost of production for a firm in coalition S is $\lambda - \mu |S|$ for some $\lambda, \mu \in \mathbb{R}_+$ such that $\lambda - 3\mu \geq 0$. Profit for firm i in coalition S under market quantity Q and firm i's quantity q_i is $(P(Q) - c_S) q_i$. Given the marginal cost profile under a coalition structure, each firm independently decides its quantity to maximize its profit. For any Nash equilibrium to be an inner solution, suppose that $\alpha - \lambda - \mu \geq 0$. For any partition π and any $i \in N$, let c_i^π be firm i's marginal cost under π. For any partition π and any $i \in N$, let q_i^π be firm i's quantity in a unique

Nash equilibrium under π. Consequently, for any partition π and any $i \in N$,

$$q_i^{\pi} = \frac{\alpha + \sum_{j \in N} c_j^{\pi}}{4} - c_i^{\pi}.$$

For any $i \in N$, $c_i^{\overline{\pi}} = \lambda - 3\mu$. Thus, for any $i \in N$,

$$q_i^{\overline{\pi}} = \frac{\alpha + 3(\lambda - 3\mu)}{4} - (\lambda - 3\mu) = \frac{\alpha - \lambda + 3\mu}{4}.$$

For any distinct $i, j \in N$, $c_i^{\pi\{j\}} = \lambda - 2\mu$ and $c_j^{\pi\{j\}} = \lambda - \mu$. Thus, for any distinct $i, j \in N$,

$$\begin{aligned} q_i^{\pi\{j\}} &= \frac{\alpha + 2(\lambda - 2\mu) + (\lambda - \mu)}{4} - (\lambda - 2\mu) = \frac{\alpha - \lambda + 3\mu}{4} \\ q_j^{\pi\{j\}} &= \frac{\alpha + 2(\lambda - 2\mu) + (\lambda - \mu)}{4} - (\lambda - \mu) = \frac{\alpha - \lambda - \mu}{4}. \end{aligned}$$

For any $i \in N$, $c_i^{\underline{\pi}} = \lambda - \mu$. Thus, for any $i \in N$,

$$q_i^{\underline{\pi}} = \frac{\alpha + 3(\lambda - \mu)}{4} - (\lambda - \mu) = \frac{\alpha - \lambda + \mu}{4}.$$

For any partition π and any $i \in N$, firm i's profit in a unique Nash equilibrium under π is equal to $(q_i^{\pi})^2$. Assume that monetary transfers are possible within each R&D coalition. Then, the sum of payoffs of firms in S in the unique Nash equilibrium of the strategic form game under π is the worth of S under π. Therefore, $\overline{v} = 3\left((\alpha - \lambda + 3\mu)/4\right)^2$, $v_2 = 2\left((\alpha - \lambda + 3\mu)/4\right)^2$, $v_1 = \left((\alpha - \lambda - \mu)/4\right)^2$ and $\underline{v} = \left((\alpha - \lambda + \mu)/4\right)^2$.

Suppose that $\mu > 0$. Then, $\overline{v} - \underline{v} - v_2 = \mu(\alpha - \lambda + 2\mu)/4 > 0$ and $\overline{v} + 3v_1 - 3v_2 = -3\mu(\alpha - \lambda + \mu)/2 < 0$. Therefore, there exists an efficient SSSPE of $G(\delta)$, while no gradual-coalition-formation one exists for sufficiently small discount factor δ, and there exists a gradual-coalition-formation SSSPE and no efficient SSSPE for sufficiently large discount factor δ. In the limit case as $\delta \to 1$, since

$$v_2 - \frac{1}{3}\overline{v} = \left(\frac{\alpha - \lambda + 3\mu}{4}\right)^2 > v_1 = \left(\frac{\alpha - \lambda - \mu}{4}\right)^2,$$

by Proposition 7, there exists a gradual-coalition-formation SSSPE and no efficient SSPE.

6 Conclusion

We investigated a noncooperative coalitional bargaining game with externalities and renegotiations. We provided a necessary and sufficient condition for an efficient SSPE of the bargaining game to exist. As the discount factor is close to one, (i) the existence of an efficient SSPE, (ii) the bargaining deviation-proofness of the NBS, and (iii) the nonemptiness of the NBC are equivalent. In a three symmetric player case, if the NBS has a bargaining deviation by some sub-coalition, the grand coalition cannot be formed in one step.

Appendix

A Lemmas

We prepare a lemma to prove propositions. Let $\delta \in [0,1)$. Let σ be an SSPE of $G(\delta)$. Let u be the payoff configuration of σ under δ.

Lemma 1. *Let (π, M) be a state. Let $I \in \pi$. Let (ρ, x) be a proposal of player $[I]^{-1}_{\pi M}$. If for any $J \in \rho \setminus \{I\}$, $x_J > (1-\delta) v^\pi_J + \delta u^\pi_J$, (ρ, x) is accepted. If for some $J \in \rho \setminus \{I\}$, $x_J < (1-\delta) v^\pi_J + \delta u^\pi_J$, (ρ, x) is rejected.*

Proof. Suppose that for any $J \in \rho \setminus \{I\}$, $x_J > (1-\delta) v^\pi_J + \delta u^\pi_J$. Suppose that (ρ, x) is rejected. Let J be the element in π such that $[J]^{-1}_{\pi M}$ is the last rejecter. The payoff of player $[J]^{-1}_{\pi M}$ by σ at the responding node is $(1-\delta) v^\pi_J + \delta u^\pi_J$. Her payoff by deviation to accepting the proposal is x_J, which is greater than the payoff by σ. This is a contradiction. Thus, (ρ, x) is accepted.

Suppose that for some $J \in \rho \setminus \{I\}$, $x_J < (1-\delta) v^\pi_J + \delta u^\pi_J$. Suppose that (ρ, x) is accepted. Let J be an element in π such that $x_J < (1-\delta) v^\pi_J + \delta u^\pi_J$. The payoff of player $[J]^{-1}_{\pi M}$ by σ at the responding node is x_J. Her payoff by deviation to accepting the proposal is $(1-\delta) v^\pi_J + \delta u^\pi_J$, which is greater than the payoff by σ. This is a contradiction. Thus, (ρ, x) is rejected. □

Lemma 2. *Let (π, M) be a state. Let $I \in \pi$. In σ, the proposal of player $i = [I]^{-1}_{\pi M}$ in σ is accepted by all responders.*

Proof. Suppose that in σ, the proposal of player $i = [I]^{-1}_{\pi M}$ in σ is rejected by some responder. Then, player i's payoff by σ at the proposing node is $(1-\delta)\, v^{\pi}_{I} + \delta u^{\pi}_{I}$. Let $\epsilon \in \mathbb{R}_{++}$. let y be an element of $\mathbb{R}^{\pi}$ such that $y^{\epsilon}_{J} = (1-\delta)\, v^{\pi}_{J} + \delta u^{\pi}_{J} + \epsilon$ for any $J \in \pi \setminus \{i\}$. Then, by the definition of y and Lemma 1, in σ, player i's proposal (π, y) is accepted by all responders. Thus, by deviating to proposal (π, y), player i obtains $v^{\{N\}}_{N} - \sum_{J \in \pi \setminus \{I\}} y_J$. Since σ is an SSPE, $(1-\delta)\, v^{\pi}_{I} + \delta u^{\pi}_{I} \geq v^{\{N\}}_{N} - \sum_{J \in \pi \setminus \{I\}} y_J$. Thus, by the definition of y, i.e., $v^{\{N\}}_{N} \leq \sum_{J \in \pi}((1-\delta)\, v^{\pi}_{J} + \delta u^{\pi}_{J}) + (|\pi| - 1)\epsilon$. Hence, since ϵ is arbitrary, $v^{\{N\}}_{N} \leq \sum_{J \in \pi}((1-\delta)\, v^{\pi}_{J} + \delta u^{\pi}_{J})$. Note that since the grand coalition is efficient in (N, v) and the transfers in summation $\sum_{J \in \pi} u^{\pi}_{J}$ are offset, $\sum_{J \in \pi} u^{\pi}_{J} \leq v^{\{N\}}_{N}$. Then, $v^{\{N\}}_{N} \leq \sum_{J \in \pi} v^{\pi}_{J}$, which contradicts to that the grand coalition is strictly efficient in (N, v). □

Lemma 3. *Let (π, M) be a state. Let $I \in \pi$. Let (ρ, x) be the proposal of player $i = [I]^{-1}_{\pi M}$ in σ at her proposing node in any round with state (π, M). Then, for any $J \in \pi \setminus \{I\}$, $x_J = (1-\delta)\, v^{\pi}_{J} + \delta u^{\pi}_{J}$.*

Proof. By Lemma 2, the proposal is accepted in σ. Thus, player i's payoff by σ at the proposing node in the any round with state (π, M) is $(1-\delta)\, v^{\pi|\rho}_{\bar{\rho}} + \delta u^{\pi|\rho}_{\bar{\rho}} - x_I$. Since player i's proposal (ρ, x) is accepted in σ, by Lemma 1, for any $J \in \rho \setminus \{I\}$, $x_J \geq (1-\delta)\, v^{\pi}_{J} + \delta u^{\pi}_{J}$. Suppose that for some $J \in \rho \setminus \{I\}$, $x_J > (1-\delta)\, v^{\pi}_{J} + \delta u^{\pi}_{J}$. Let $\epsilon := (x_J - (1-\delta)\, v^{\pi}_{J} + \delta u^{\pi}_{J})/2 > 0$ and y be an element in $\mathbb{R}^{\pi}$ such that $y_K = x_K + \epsilon/\,(|\rho| - 1)$ for any $K \in \pi \setminus \{J\}$ and $y_J = x_J - \epsilon$. Then, for any $J \in \pi \setminus \{I\}$, $y_J > (1-\delta)\, v^{\pi}_{J} + \delta u^{\pi}_{J}$. Thus, by Lemma 1, player i's proposal (ρ, y) is accepted in σ. Note that since the coalition proposed in the deviation is the same that proposed in σ, the deviation does not change the state in the next round, and thus, it does not change player i's expected payoff in the subgame following the deviation. Then, player i's payoff by the deviation to proposing (ρ, y) at the proposing node is $(1-\delta)\, v^{\pi|\rho}_{\bar{\rho}} + \delta u^{\pi|\rho}_{\bar{\rho}} + y_I = (1-\delta)\, v^{\pi|\rho}_{\bar{\rho}} + \delta u^{\pi|\rho}_{\bar{\rho}} + x_I + \epsilon > v^{\{N\}}_{N} + x_I$, which is her payoff by σ. This is a contradiction. □

B Proof of Proposition 1

Let $\delta \in [0, 1)$. Let σ be an efficient SSPE of $G(\delta)$. Let u be the payoff configuration of σ under δ. Let (π, M) be a state.

Let $I \in \pi$. Let (ρ, x) be the proposal of player $i := [I]_{\pi M}^{-1}$ by σ in any round with state (π, M). Since σ is efficient, $\rho = \pi$. By Lemma 3, for any $J \in \pi \backslash \{I\}$, $x_J = (1-\delta)\, v_J^\pi + \delta u_J^\pi$.

Let $I \in \pi$. By the argument above,

$$\begin{aligned} u_I^\pi &= \frac{1}{|\pi|}\left(v_N^{\{N\}} - \sum_{J \in \pi \backslash \{I\}} \left((1-\delta)\, v_J^\pi + \delta u_J^\pi \right) \right) + \frac{|\pi| - 1}{|\pi|} \left((1-\delta)\, v_I^\pi + \delta u_I^\pi \right) \\ &= \frac{1}{|\pi|}\left(v_N^{\{N\}} - \sum_{J \in \pi} \left((1-\delta)\, v_J^\pi + \delta u_J^\pi \right) \right) + \left((1-\delta)\, v_I^\pi + \delta u_I^\pi \right). \end{aligned}$$

Hence, since $\sum_{J \in \pi} u_J^\pi = v_N^{\{N\}}$,

$$(1-\delta)\, u_I^\pi = \frac{1}{|\pi|}\left((1-\delta)\, v_N^{\{N\}} - \sum_{J \in \pi} (1-\delta)\, v_J^\pi \right) + (1-\delta)\, v_I^\pi.$$

Therefore, since $\delta < 1$,

$$u_I^\pi = \frac{1}{|\pi|}\left(v_N^{\{N\}} - \sum_{J \in \pi} v_J^\pi \right) + v_I^\pi = b_I^\pi.$$

By the argument above, for any $I \in \pi$, player $[I]_{\pi M}^{-1}$ proposes $\left(\pi, x^I\right)$ such that for any $J \in \pi \setminus \{I\}$, $x_J^I = (1-\delta)\, v_J^\pi + \delta u_J^\pi = \delta \left(v_N^{\{N\}} - \sum_{K \in \pi} v_K^\pi \right) / |\pi| + v_J^\pi$. □

C Proof of Proposition 2

Let $\delta \in [0, 1)$.

Necessity Suppose that there exists an efficient SSPE σ of $G(\delta)$. Let u be the payoff configuration of σ. Let $\pi \in \Pi_N$ and $\rho \in 2^\pi \setminus \{\emptyset\}$. Let M be a complete system of representatives of π. Let $I \in \rho$. By Lemma 3, in a round with state (π, M), player $i = [I]_{\pi M}^{-1}$ proposes (π, x) such that for any $J \in \pi \setminus \{I\}$, $x_J = (1-\delta)\, v_J^\pi + \delta u_J^\pi$. Since σ is efficient, player i's payoff of σ conditional on being a proposal in in a round with state

(π, M) is

$$v_N^{\{N\}} - \sum_{J\in\pi\setminus\{I\}} \left((1-\delta)\, v_J^\pi + \delta u_J^\pi\right). \tag{9}$$

Let $\epsilon \in \mathbb{R}_{++}$. Let y be the element in $\mathbb{R}^\rho$ such that for any $J \in \rho \setminus \{I\}$, $y_J = (1-\delta)\, v_J^\pi + \delta u_J^\pi + \epsilon$ and $y_I = (1-\delta)\, v_{\bar{\rho}}^{\pi|\rho} + \delta u_{\bar{\rho}}^{\pi|\rho} - \sum_{J\in\rho\setminus\{I\}} y_J$. Then, by Lemma 1, player i's proposal (ρ, y) in a round with state (π, M) is accepted by all responders. Thus, by one-stage deviation to proposing (ρ, y), player i obtains

$$(1-\delta)\, v_{\bar{\rho}}^{\pi|\rho} + \delta u_{\bar{\rho}}^{\pi|\rho} - \sum_{J\in\rho\setminus\{I\}} \left((1-\delta)\, v_J^\pi + \delta u_J^\pi + \epsilon\right). \tag{10}$$

Since σ is an SSPE, (9) is greater than or equal to (10). Since ϵ is arbitrary and (10) is decreasing in ϵ,

$$v_N^{\{N\}} - \sum_{J\in\pi\setminus\{I\}} \left((1-\delta)\, v_J^\pi + \delta u_J^\pi\right) \geq (1-\delta)\, v_{\bar{\rho}}^{\pi|\rho} + \delta u_{\bar{\rho}}^{\pi|\rho} - \sum_{J\in\rho\setminus\{I\}} \left((1-\delta)\, v_J^\pi + \delta u_J^\pi\right).$$

Note that by Proposition 1, for any $(J, \tau) \in \mathcal{C}$, $u_J^\tau = b_J^\tau$. Then, substituting $u_J^\pi = b_J^\pi$ for any $J \in \pi$ and $u_{\bar{\rho}}^{\pi|\rho} = b_{\bar{\rho}}^{\pi|\rho}$, we have (1).

Sufficiency Suppose that for any $\pi \in \Pi_N$ and any $\rho \in 2^\pi \setminus \{\emptyset\}$, (1) holds. Define the strategy profile σ as follows: in any round with state each (π, M), each player $[I]_{\pi M}^{-1}$ proposes $\left(\pi, x^I\right)$ such that for any $J \in \pi \setminus \{I\}$,

$$x_J^I = v_J^\pi + \frac{1}{|\pi|}\delta \left(v_N^{\{N\}} - \sum_{K\in\pi} v_K^\pi\right);$$

she accepts any proposal (ρ, y) if and only if $y_I \geq v_I^\pi + \delta \left(v_N^{\{N\}} - \sum_{J\in\pi} v_J^\pi\right) / |\pi|$. Note that σ satisfies the stationarity property. Let u be the payoff configuration of σ. Since in σ, every proposal in σ is accepted by all responders, for any $(I, \pi) \in \mathcal{C}$,

$$\begin{aligned} u_I^\pi = & \frac{1}{|\pi|}\left(v_N^{\{N\}} - \sum_{J \in \pi \setminus \{I\}}\left(v_J^\pi + \delta \frac{1}{|\pi|}\left(v_N^{\{N\}} - \sum_{J \in \pi} v_J^\pi\right)\right)\right) \\ & + \frac{|\pi| - 1}{|\pi|}\left(v_I^\pi + \delta \frac{1}{|\pi|}\left(v_N^{\{N\}} - \sum_{J \in \pi} v_J^\pi\right)\right) \\ = & \, v_I^\pi + \frac{1}{|\pi|}\left(v_N^{\{N\}} - \sum_{J \in \pi} v_J^\pi\right) \end{aligned}$$

If player $[I]_{\pi M}^{-1}$ proposes $\left(\pi, x^I\right)$, the proposal is accepted by all responders in σ, she receives payoff

$$v_N^{\{N\}} - \sum_{J \in \pi \setminus \{I\}}\left(v_J^\pi + \delta \frac{1}{|\pi|}\left(v_N^{\{N\}} - \sum_{J \in \pi} v_J^\pi\right)\right). \tag{11}$$

On the other hand, if player $[I]_{\pi M}^{-1}$ proposes an arbitrary proposal (ρ, x) to be accepted by all responders in σ, by Lemma 1, she obtains

$$(1 - \delta) v_{\bar{\rho}}^{\pi|\rho} + \delta u_{\bar{\rho}}^{\pi|\rho} - \sum_{J \in \rho \setminus \{I\}}\left(v_J^\pi + \frac{1}{|\pi|}\delta\left(v_N^{\{N\}} - \sum_{J \in \pi} v_J^\pi\right)\right). \tag{12}$$

at most. Condition (1) implies that (11) is greater than or equal to (12). Moreover, if player $[I]_{\pi M}^{-1}$ proposes an unacceptable proposal, she obtains

$$(1 - \delta) v_I^\pi + \delta u_I^\pi = v_I^\pi + \frac{1}{|\pi|}\delta\left(v_N^{\{N\}} - \sum_{J \in \pi} v_J^\pi\right). \tag{13}$$

Since $v_N^{\{N\}} \geq \sum_{J \in \pi} v_J^\pi$, (11) is greater than or equal to (13). Thus, σ is a local optimal strategy for the proposer $[I]_{\pi M}^{-1}$. Moreover, each responder $[J]_{\pi M}^{-1}$ obtains

$$v_J^\pi + \delta u_J^\pi = v_J^\pi + \frac{1}{|\pi|}\delta\left(v_N^{\{N\}} - \sum_{J \in \pi} v_J^\pi\right)$$

if he rejects a proposal. Therefore, σ prescribes a local optimal strategy for the responder j. By the one-stage deviation principle, σ is a subgame perfect equilibrium. It is clear that σ is efficient. Therefore, σ is an efficient SSPE of $G(\delta)$. $\square$

D Proof of Proposition 3

(i) $\Longleftrightarrow$ (ii) Note that the left hand sides of (1) are decreasing and the right hand side is increasing in δ. Therefore, by Proposition 2, for some $\bar{\delta} \in [0,1)$, for any $\delta \in \left[\bar{\delta}, 1\right)$, there exists an efficient SSPE of $G(\delta)$ if and only if for any $\pi \in \Pi_N$ and any $\rho \in 2^\pi \setminus \{\emptyset\}$,

$$|\rho| \frac{v_N^{\{N\}} - \sum_{I \in \pi} v_I^\pi}{|\pi|} + \sum_{I \in \rho} v_I^\pi \geq \frac{v_N^{\{N\}} - \sum_{I \in \pi|\rho} v_I^{\pi|\rho}}{|\pi| - |\rho| + 1} + v_{\bar{\rho}}^{\pi|\rho}.$$

Since this inequality is equivalent to $\sum_{I \in \rho} b_I^\pi \geq b_{\bar{\rho}}^{\pi|\rho}$, (i) is equivalent to (ii).

(ii) $\Longrightarrow$ (iii) Suppose that (ii) holds. Let $\pi \in \Pi_N$. Choose the bargaining deviation-proof NBS b^π under π. Suppose that b^π is not in the NBC under π. Then, for some $\rho \in 2^\pi \setminus \{\emptyset\}$, there exists $\sum_{I \in \rho} b_I^\pi < V_\rho^\pi$. Thus, by the definition of V^π, $\sum_{I \in \rho} b_I^\rho < b_{\bar{\rho}}^{\pi|\rho}$. This is a contradiction to the fact that b^π is a bargaining deviation-proof NBS. Hence, b^π is in the the NBC under π. Therefore, the NBC under π is nonempty.

(iii) $\Longrightarrow$ (ii) For any $\pi \in \Pi_N$. let b^π be the NBS under π. Suppose that (iii) holds. Let $\pi \in \Pi_N$. By the supposition, there exists an allocation x in the NBC under π. Then, for any $I \in \pi$, $\sum_{J \in \{I\}} x_J \geq V_{\{I\}}^\pi$, i.e., $x_I \geq b_{\{\bar{I}\}}^{\pi|\{I\}} = b_I^\pi$. Since x is feasible, $\sum_{I \in \pi} x_I \leq V_\pi^\pi = b_{\bar{\pi}}^{\pi|\pi} = b_N^{\{N\}} = v_N^{\{N\}} = \sum_{I \in \pi} b_I^\pi$. Thus, for any $I \in \pi$, $x_I^\pi = b_I^\pi$. Thus, b^π is in the NBC under π. Thus, for any $\rho \in 2^\pi \setminus \{\emptyset\}$, $\sum_{I \in \rho} b_I^\pi \geq V_\rho^\pi = b_{\bar{\rho}}^{\pi|\rho}$. This implies that b^π is a bargaining deviation-proof NBS. $\square$

E Proof of Proposition 5

Let u be the payoff configuration of σ. Obviously, $u_N^{\{N\}} = \overline{v}$. Since any subgame starting with state (π, M) such that $|\pi| = 2$ is a Binmore-Rubinstein bilateral bargaining game, for any $I \in \underline{\pi}$, we have that $u_I^{\pi_I} = (\overline{v} + v_1 - v_2)/2$ and $u_{N \setminus I}^{\pi_I} = (\overline{v} - v_1 + v_2)/2$. Since σ is symmetric, for some generator g such that $\underline{\pi}$ is a cyclic group with g, in σ, for any

$I \in \underline{\pi}$, player $[I]_{\pi N}^{-1}$ proposes coalition $\{I, Ig\}$ in the first round. Thus, by Lemmas 2 and 3, for any $I \in \underline{\pi}$,

$$u_I^{\pi} = \frac{1}{3}\left((1-\delta)\, v_2 - (1-\delta)\, \underline{v} - \delta u_{Ig}^{\pi} + \delta u_{N\setminus\{Ig^2\}}^{\pi_{Ig^2}}\right) + \frac{1}{3}((1-\delta)\, \underline{v} + \delta u_I^{\pi}) \\ + \frac{1}{3}\left((1-\delta)\, v_1 + \delta u_I^{\pi_I}\right).$$

Substituting $u_I^{\pi_I} = (\overline{v} + v_1 - v_2)/2$ and $u_{N\setminus\{Ig^2\}}^{\pi_{Ig^2}} = (\overline{v} - v_1 + v_2)/2$ and solving the above 3-dimensional system of simultaneous equations, then, for any $I \in \underline{\pi}$, we have that $u_I^{\pi} = (\delta\overline{v} + (1-\delta)\, v_1 + (1-\delta)\, v_2)/3$. Therefore, $u = u^{\delta}$. □

F Proof of Proposition 6

Since δ is fixed in this proof, we omit δ of u^{δ}.

Necessity Suppose that there exists a gradual-coalition-formation SSSPE σ. By Proposition 5, the payoff configuration of σ is u. Let $I \in \underline{\pi}$. Let $\{I, J\}$ be the coalition proposed by player $i := [I]_{\pi N}^{-1}$ in the first round in σ. Let $K \in \underline{\pi} \setminus \{I, J\}$. Then, by Lemma 3, player i's payoff by σ conditional on being a proposer in the first round is $(1-\delta)\, v_2 + \delta u_{N\setminus\{K\}}^{\pi_K} - ((1-\delta)\, \underline{v} + \delta u_J^{\pi})$. Let $\epsilon \in \mathbb{R}_{++}$. Let y be an element of $\mathbb{R}^{\underline{\pi}}$ such that for any $L \in \underline{\pi} \setminus \{I\}$, $y_L = (1-\delta)\, \underline{v} + \delta u_L^{\pi} + \epsilon$. Then, by deviating to proposal $(\underline{\pi}, y)$, which is accepted by all responders by Lemma 1, player i obtains $\overline{v} - (1-\delta)\, \underline{v} - \delta u_J^{\pi} - (1-\delta)\, \underline{v} - \delta u_K^{\pi} - 2\epsilon$. Since σ is an SSPE, $(1-\delta)\, v_2 + \delta u_{N\setminus\{K\}}^{\pi_K} - ((1-\delta)\, \underline{v} + \delta u_J^{\pi}) \geq \overline{v} - (1-\delta)\, \underline{v} - \delta u_J^{\pi} - (1-\delta)\, \underline{v} - \delta u_K^{\pi} - 2\epsilon$. Thus, since $u_I^{\pi} = u_J^{\pi} = u_K^{\pi}$, $(1-\delta)\, v_2 + \delta u_{N\setminus\{I\}}^{\pi_I} - ((1-\delta)\, \underline{v} + \delta u_I^{\pi}) \geq \overline{v} - 2\left((1-\delta)\, \underline{v} + \delta u_I^{\pi}\right) - 2\epsilon$. Note that ϵ is an arbitrary positive number. Then, $(1-\delta)\, v_2 + \delta u_{N\setminus\{I\}}^{\pi_I} - ((1-\delta)\, \underline{v} + \delta u_I^{\pi}) \geq \overline{v} - 2\left((1-\delta)\, \underline{v} + \delta u_I^{\pi}\right)$. Substituting (6) into the above inequality, we have (7).

Sufficiency We can see that the following strategy profile σ is an SSPE of $G(\delta)$ under the above condition:

(i) In state $(\underline{\pi}, N)$, for any $I, J, K \in \underline{\pi}$ such that $J = Ig$ and $K = Ig^2$, player $[I]_{\pi N}^{-1}$ proposes coalition $(\{I, J\}, x)$ such that $x_J = (1-\delta)\, v_J^{\pi} + \delta u_J^{\pi}$, and she accepts the proposal (ρ, x) if and only if $x_I \geq (1-\delta)\, v_{\underline{I}}^{\pi} + \delta u_{\underline{I}}^{\pi}$.

(ii) In state (π, M) with $|\pi| = 2$, for any distinct $I, J \in \pi$, player $[I]^{-1}_{\pi_K M}$ proposes (π, x) such that $x_J = (1-\delta)\, v_J^{\pi} + \delta u_J^{\pi}$, and she accepts the proposal x_I if and only if $x_I \geq (1-\delta)\, v_I^{\pi} + \delta u_I^{\pi}$.

For any state (π, M) and any $I \in \pi$, the expected payoff of player $[I]^{-1}_{\pi M}$ by σ in any subgame starting with state (π, M) is u_I^{π}. The bargaining game in states where two players are active is same as a Binmore-Rubinstein bilateral bargaining game. In σ, the proposer offers just the continuation payoff for the opponent and the responder's threshold for acceptance or rejection is her continuation payoff. Thus, any action by σ is optimal in this state.

Let us move on to the state with three active players. Let $I \in \underline{\pi}$. Since the continuation payoff when the responder $[I]^{-1}_{\pi N}$ rejects the proposal is $(1-\delta)\, v_{\underline{I}}^{\pi} + \delta u_{\underline{I}}^{\pi}$, it is optimal for the responder to take the above action. The payoff of player $[I]^{-1}_{\pi N}$ by σ conditional on being a proposer in the first round is

$$(1-\delta)\, v_2 + \delta u_{N\setminus\{Ig^2\}}^{\pi_{Ig^2}} - \left((1-\delta)\underline{v} + \delta u_{\underline{Ig}}^{\pi}\right) = (1-\delta)\, v_2 + \delta u_{N\setminus\{I\}}^{\pi_I} - \left((1-\delta)\underline{v} + \delta u_{\underline{I}}^{\pi}\right).$$

For any distinct $J, K \in \underline{\pi} \setminus \{I\}$, if player $[I]^{-1}_{\pi N}$ proposes a two-player coalition $\{I, J\}$ to be accepted, she obtains

$$(1-\delta)\, v_2 + \delta u_{N\setminus\{K\}}^{\pi_K} - \left((1-\delta)\, \underline{v} + \delta u_{\underline{J}}^{\pi}\right) = (1-\delta)\, v_2 + \delta u_{N\setminus\{I\}}^{\pi_I} - \left((1-\delta)\, \underline{v} + \delta u_{\underline{I}}^{\pi}\right)$$

at most. Moreover, if she proposes the grand coalition to be accepted, she obtains

$$\overline{v} - \sum_{J \in \underline{\pi}\setminus\{I\}} \left((1-\delta)\, \underline{v} + \delta u_{\underline{J}}^{\pi}\right) = \overline{v} - 2\left((1-\delta)\, \underline{v} + \delta u_{\underline{I}}^{\pi}\right)$$

at most. On the other hand, if she offers an unacceptable proposal, she obtains

$$(1-\delta)\, \underline{v} + \delta u_{\underline{I}}^{\pi}.$$

By substituting (6) to condition (7), we have

$$(1-\delta)v_2 + \delta u_{N\setminus I}^{\pi_I} - ((1-\delta)\underline{v} + \delta u_I^{\pi}) \geq \overline{v} - 2\left((1-\delta)\,\underline{v} + \delta u_I^{\pi}\right).$$

This implies that the proposer in the initial state $(\underline{\pi}, N)$ has no incentive to deviate from the above strategies. Thus, the above strategy profile is an SSPE of $G(\delta)$. □

G Proof of Proposition 7

Let R be the right hand side of (4). Let L be a function from $\mathbb{R}$ to $\mathbb{R}$ such that for any $\delta \in \mathbb{R}$, $L(\delta)$ is equal to the left hand side of (7).

(i) Suppose that $v_2 - \overline{v}/3 > v_1$. Then, $L(1) = v_2 - \overline{v}/3 - v_1 > 0$. Thus, for some $\bar{\delta} \in [0,1)$, for any $\delta \in [\bar{\delta}, 1)$, $L(\delta) \geq 0$, and thus, by Proposition 6, there exists a gradual-coalition-formation SSSPE of $G(\delta)$. Let $\hat{\delta} := (R+1)/2$. By the supposition, $0 \leq R < 1$. Thus, $\hat{\delta} \in [0,1)$. For any $\delta \in \left[\hat{\delta}, 1\right)$, $\delta > R$, and thus, there exists no efficient SSPE of $G(\delta)$. By redefining $\bar{\delta}$ as $\max\left\{\bar{\delta}, \hat{\delta}\right\}$, we have the conclusion.

(ii) Suppose that $v_2 - \overline{v}/3 \leq v_1$. Then, $R \geq 1$. Thus, for any $\delta \in [0,1)$, (4) holds, and thus, there exists an efficient SSPE of $G(\delta)$. By the strict superadditivity, $L'(1) = (4/3)(\overline{v} - v_2 - v_1) + (1/3)(\overline{v} - 3\underline{v}) > 0$. Thus, for some $\bar{\delta} \in [0,1)$, L is strictly increasing on $[\bar{\delta}, 1)$. Hence, for any $\delta \in [\bar{\delta}, 1)$, $L(\delta) < L(1)$. $L(1)$ is less than or equal to 0 by the supposition. By Proposition 6, there exists no gradual-coalition-formation SSSPE of $G(\delta)$. □

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