Note on the Equal Split Solution in an n-Person Noncooperative Bargaining Game

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Abstract

This note examines a noncooperative bargaining game model to implement the "equal split" solution in a transferable utility coalitional form game provided by Hart and Mas-Colell (*Econometrica* **64**, 1996). We first clarify the relationship between the equal split solution and the Nash bargaining solution in a coalitional form game and extend the model to a nontransferable utility coalitional form game. We then provide a sufficient condition for generating the Nash bargaining solution configuration and the equal split solution as the limit of the stationary subgame-perfect equilibrium payoffs of Hart and Mas-Colell's bargaining game when the probabilities of the breakdown of negotiations converge to zero.

JEL Classification Numbers: C72, C78.

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1 Introduction

Hart and Mas-Colell (1996) provide a noncooperative bargaining model to implement the "equal split" solution in the limit of stationary subgame-perfect equilibria (SSPEs) as the cost of delay goes to zero[1]. The "equal split" solution assigns the equal division $v(S)/|S|$ of the worth $v(S)$ to the member of a coalition $S \subset N = \{1, 2, \ldots, n\}$ in a coalitional form game (N, v) with transferable utility (a TU game). The solution is very interesting because the solution for each coalition S is not sensitive to the worth of subcoalitions. In addition, we note that the equal split solution is coincident with the Nash bargaining solution payoff configuration (which is called, in this paper, the *subcoalition-consistent Nash bargaining solution*) in the case of a TU game. It is easy to see that the Nash bargaining solution of the bargaining problem $(v(S), r_S)$, where the disagreement point r_S is the origin, becomes the equal division $v(S)/|S|$.

In this note, we investigate Hart and Mas-Colell's (1996) bargaining procedure. This bargaining procedure follows the tradition in setting up a sequential, alternating-offer, perfect information game such as in Binmore, et al. (1986) and Rubinstein (1982). In each round, a player is selected as a proposer with equal probability among all existing players. The proposer can propose a feasible payoff vector for the active players. The requirement for agreement is unanimity. The key feature of the bargaining procedure is the rule regarding what happens if some player rejects the proposal. The active players are faced with the risk of the breakdown of negotiations. Negotiations among the same members continue to the next round with almost

[1]Hart and Mas-Colell's (1996) paper has been known as a provision of a noncooperative bargaining model to generate the Shapley value. In Section 6 of their paper, they generalized the bargaining procedure and obtained the equal split solution in equilibrium.

all probability, but, with a small probability, one of the responders drops out (this is a partial breakdown) and the game goes to the next round with the remaining players. We extend the bargaining situation to an n-person coalitional form game with non-transferable utility (an NTU game).

We show that the bargaining procedure does not necessarily yield either the subcoalition-consistent Nash bargaining solution or the equal split solution as an SSPE outcome in a general NTU game when the probability of partial breakdown goes to zero. More concretely, we give an example in which the subcoalition-consistent Nash bargaining solution is unable to be the limit of SSPE payoff configurations in the noncooperative bargaining game as the breakdown probability goes to zero. We then provide a sufficient condition to implement the subcoalition-consistent Nash bargaining solution through the noncooperative bargaining procedure. That condition of coalitional form games is so called the *Proportionality condition*. The Proportionality condition is satisfied by a class of TU and hyperplane coalitional form games with a common weight. We also show that in the bargaining over the division of a pie, if all players have the same risk attitudes, then the corresponding coalitional form game including NTU games satisfies the Proportionality condition. The equal split solution is implemented by our noncooperative bargaining game if the coalitional form game follows the Proportionality condition and, in addition, the set of feasible payoff allocations in the two-person bargaining is bilaterally symmetric. A TU game, considered in Hart and Mas-Colell, is satisfying the both conditions by chance.

The related literatures concerning the noncooperative foundation of the n-person Nash bargaining solution is as follows. To start with, Krishna and Serrano (1996) provided a distinct noncooperative bargaining game to yield the n-person Nash bargaining solution as an equilibrium agreement when players are patient. However, their bargaining game did not require unanimous agreement among players and considered only a pure bargaining problem. As shown in Hart and Mas-Colell (1996), the Nash bargaining solution is realized as the limit of SSPE payoff vectors of our noncooperative bargaining game if the bargaining situation is described by a pure bargaining problem. Okada (2005) also presented a noncooperative foundation of the asymmetric Nash bargaining solution for a general cooperative game in strategic form. His cooperative game in strategic form contains the bargaining situations in this paper as special cases. However, while Okada's (2005) model allows players to form coalitions, our model does not allow strategies of coalition formation for each player. As a result, the sufficient condition for implementing the Nash bargaining solution in our model is quite different.

This paper is organized as follows. In Section 2, we provide the noncooperative bargaining model and derive the basic equations that are satisfied by the equilibrium payoff configuration in an NTU game. In Section 3, we introduce the concept of the subcoalition-consistent Nash bargaining solution and provide a simple example in which the subcoalition-consistent Nash bargaining solution is not generated by our noncooperative bargaining game in an NTU game. Section 4 presents a sufficient condition for implementing the subgame-consistent Nash bargaining solution and the equal split solution. The proof of the Theorem 1 is given in the Appendix.

2 Extension to NTU case

The bargaining situation is described by an n-person coalitional form game (N, V). $N = \{1, 2, \ldots, n\}$ is a finite set of players and $V(\cdot)$ is a function that assigns a subset $V(S)$ of $\mathbb{R}^S$ to every coalition $S \subset N$. The set $V(S)$ can be interpreted as the set of all feasible payoff vectors to the members of S if they jointly commit to a certain course of action. We impose on (N, V) the following assumptions.

Assumption 1. (i) For each coalition S, the set $V(S)$ is closed, convex and comprehensive (i.e., $V(S) - \mathbb{R}^S_+ \subset V(S)$). Moreover, $V(S) \cap \mathbb{R}^S_+$ is bounded. (ii) For each coalition S, the boundary of $V(S)$, $\partial V(S)$ is smooth (i.e., at each $y \in \partial V(S)$, there exists a single outward normal direction) and nonlevel (i.e., the outward normal vector at any point of $\partial V(S)$ is positive in all coordinates). (iii) (Monotonicity) $V(S) \times \{0^{T \setminus S}\} \subset V(T)$ whenever $S \subset N$. (iv) (0-normalization) $r_i = \max\{c : c \in V(\{i\})\} = 0$.

Exception (iv), these assumptions are also made by Hart and Mas-Colell (1996). The game (N, V) is called a nontransferable utility (NTU) game. If there exist a real-valued function v such that $V(S) = \{c \in \mathbb{R}^S | \ \sum_{i \in S} c^i \leq v(S)\}$ for all $S \subset N$, the coalitional form game is called a TU (transferable utility) game. Usually, a TU game is represented by (N, v). In this case, the number $v(S)$ may be interpreted as a sum of money that the members of S can distribute among themselves.

We consider the following noncooperative bargaining procedure. Let $0 \leq \rho < 1$ be a fixed parameter.

(i) At every round $t = 1, 2, \ldots$, one player is selected as a proposer with equal probability among all players still active in bargaining. Let N^t denote the set of all active players at round t. The bargaining starts with all members of N, i.e., $N^1 = N$. The selected player i proposes a payoff vector y in $V(N^t)$.

(ii) All other players in N^t either accept or reject the proposal sequentially. We assume that the responses are made according to a predetermined order over N^t. If all other players in N^t accept, then the game ends with these payoffs. If some players reject, then the game moves to next round where, with probability ρ, the set of active players at round $t+1$ is unchanged, i.e., $N^{t+1} = N^t$, and with probability $1-\rho$, one player j drops out with equal probability among all responders in $N^t \backslash i$, and the set of active players becomes $N^{t+1} = N^t \backslash j$. The player who dropps out receives a payoff of zero.

Our bargaining model can be represented as an extensive form game with perfect information and with chance moves. We denote by $G^S(\rho)$ the bargaining model with active players S for every coalition $S \subset N$ and for a parameter ρ. We shall apply the solution concept of a *stationary subgame-perfect equilibrium* (SSPE) to our bargaining model. In order to avoid the problem of multiplicity of subgame-perfect equilibria, the concept of an SSPE is commonly adopted in the literature on the noncooperative multilateral bargaining model (see Chatterjee et al. 1993, Hart and Mas-Colell, 1996, Okada, 1996 and 2005, among others). For an SSPE strategy combination σ of $G^N(\rho)$ and every coalition $S \subset N$, let $v_S(\rho) = (v_S^i(\rho))_{i \in S} \in \mathbb{R}_+^S$ be the expected payoff vector of players for σ in the subgame $G^S(\rho)$. We call the collection $\{v_S(\rho) \mid S \subset N\}$ the *payoff configuration* of the SSPE σ of $G^N(\rho)$.

If the game is a TU game (N, v), the noncooperative bargaining procedure we consider would yield the "equal split" solution $v_S^i(\rho) = v(S)/|S|$ for all $S \subset N$ and all $i \in S$ in an SSPE of $G^N(\rho)$ as $\rho \to 1$. This is indicated in Hart and Mas-Colell (1996). Extending to the general NTU case, we obtain the following theorem.

Theorem 1. *Let (N, V) be an NTU game. For each ρ, there exists an SSPE of $G^N(\rho)$. Moreover, as $\rho \to 1$, the every limit point of SSPE payoff* *configurations,* $\{v_S | S \subset N\}$*, satisfies for all* $S \subset N$ *and all* $i \in S \subset N$*,*

$$\left[\frac{1}{|S|}\sum_{j\in S}\lambda_S^j v_S^j - \lambda_S^i v_S^i\right] + \sum_{k\in S\setminus i}\frac{1}{|S|}\left[\lambda_S^i v_{S\setminus k}^i - \frac{1}{|S|-1}\sum_{j\in S\setminus k}\lambda_S^j v_{S\setminus k}^j\right] = 0, \tag{1}$$

where $(\lambda_S^j)_{j\in S}$ *is the unique supporting normal to the boundary of* $V(S)$ *at* v_S*.*

Theorem 1 can be proved in the same line as the proofs of Proposition 6, 7, and 8 in Hart and Mas-Colell (1996). While this proof could be omitted, we provide it in the Appendix to ensure that the paper is self-contained.

3 Subcoalition-Consistent Nash bargaining solution

The Nash bargaining solution is generally defined on the bargaining problem (V, r), where V is a feasible set of payoffs and r is a disagreement point. Let us define the concept of the Nash bargaining solution for a coalitional form game (N, V).

Definition 1. Suppose that the configuration of disagreement points $\{r_S \in \mathbb{R}_+^S | S \subset N\}$ is given. The *subcoalition-consistent Nash bargaining solution* of the coalitional form game (N, V) is a function that assigns to each bargaining problem $(V(S), r_S)$ constructed by (N, V) a solution of the maximization problem:

$$\max_{v_S} \prod_{i\in N}(v_S^i - r_S^i) \text{ subject to } v_S \in V(S).$$

If the game is a TU game (N, v) and $r_S = 0$ for all $S \subset N$, the subcoalition-consistent Nash bargaining solution is the "equal split" solution; $v^i_S = v(S)/|S|$ for all S and all $i \in S$. Therefore, as a result of Hart and Mas-Colell (1996), we can claim that our bargaining procedure does indeed yield the subcoalition-consistent Nash bargaining solution of the coalitional form game (N, V) as $\rho \to 1$ if the game is a TU game. This would not extend to the general NTU game.

Example: Let us consider the bargaining over the division of a pie among three players $N = \{1, 2, 3\}$. Assume that a payoff function for player $1, 2$ is given by $u_1(x) = u_2(x) = x^{1/2}$ and that for player 3 is $u_3(x) = x^{1/3}$. The size of pie for a single player $i = 1, 2, 3$ is zero; $I(\{i\}) = 0$, and for coalitions of two players, $I(\{1, 2\}) = 8$ and $I(\{2, 3\}) = I(\{1, 3\}) = 5$. Moreover, $I(\{1, 2, 3\}) = 24$. The set of feasible allocation for a coalition S is defined by

$$X^S = \{(x_i)_{i \in S} | \sum_{i \in S} x_i \leq I(S), x_i \in \mathbb{R}_+, \forall i \in S\}.$$

Let define the characteristic function as follows:

$V(\{i\}) = \{0\}$ for $i = 1, 2, 3$,

$V(\{i, j\}) = \{(v^i, v^j) \in \mathbb{R}^2_+ \mid \exists (x_i, x_j) \in X_{\{i,j\}}, v^i \leq u_i(x_i)$ and $v^j \leq u_j(x_j)\}$,

$V(\{1, 2, 3\}) = \{(v^1, v^2, v^3) \in \mathbb{R}^3_+ | \exists (x_1, x_2, x_3) \in X_{\{1,2,3\}}, \forall i \in \{1, 2, 3\}, v^i \leq u_i(x_i)\}$.

This is an NTU game. Under the configuration of disagreement points such that $r^i_{\{i\}} = 0$ for all $i = 1, 2, 3$, $r_{\{i,j\}} = (0, 0)$ for all $i, j = 1, 2, 3$, and $r_{\{1,2,3\}} = (0, 0, 0)$, we can calculate the subcoalition-consistent Nash bargaining solution of (N, V). The Nash bargaining solution of the bargaining problem $(V(\{1, 2\}), r_{\{1,2\}})$ is $a^1_{\{1,2\}} = u_1(4) = 4^{1/2} = 2$ and $a^2_{\{1,2\}} = u_2(4) = 4^{1/2} = 2$. The Nash bargaining solution of $(V(\{1, 3\}), r_{\{1,3\}})$ is $a^1_{\{1,3\}} = u_1(3) = 3^{1/2}$ and $a^3_{\{1,3\}} = u_3(2) = 2^{1/3}$. Similarly, that of $(V(\{2, 3\}), r_{\{2,3\}})$

is $a^2_{\{2,3\}} = u_2(3) = 3^{1/2}$ and $a^3_{\{2,3\}} = u_3(2) = 2^{1/3}$. For the bargaining problem $(V(\{1,2,3\}), r_{\{1,2,3\}})$, the Nash bargaining solution is given by $a^1_{\{1,2,3\}} = a^2_{\{1,2,3\}} = u_1(9) = u_2(9) = 9^{1/2} = 3$ and $a^3_{\{1,2,3\}} = u_3(6) = 6^{1/3}$.

If $N = \{1,2,3\}$, the basic equations (1) in Theorem 1, which is satisfied by the limit of SSPE payoff configurations, are reduced to

$$
\begin{aligned}
& v^i_{\{i\}} = 0 \text{ for } i = 1,2,3, \\
& \lambda^i_{\{i,j\}} v^i_{\{i,j\}} = \lambda^j_{\{i,j\}} v^j_{\{i,j\}} \text{ for } i,j = 1,2,3, i \neq j, \\
& \lambda^1_{\{1,2,3\}}(v^1_{\{1,2,3\}} - \frac{1}{2} v^1_{\{1,2\}}) = \lambda^2_{\{1,2,3\}}(v^2_{\{1,2,3\}} - \frac{1}{2} v^2_{\{1,2\}}), \\
& \lambda^1_{\{1,2,3\}}(v^1_{\{1,2,3\}} - \frac{1}{2} v^1_{\{1,3\}}) = \lambda^3_{\{1,2,3\}}(v^3_{\{1,2,3\}} - \frac{1}{2} v^3_{\{1,3\}}), \\
& \lambda^2_{\{1,2,3\}}(v^2_{\{1,2,3\}} - \frac{1}{2} v^2_{\{2,3\}}) = \lambda^3_{\{1,2,3\}}(v^3_{\{1,2,3\}} - \frac{1}{2} v^3_{\{2,3\}}).
\end{aligned}
$$

Note that, by definition of the characteristic function, the vector $(\lambda^i_{\{i,j\}}, \lambda^j_{\{i,j\}})$ is proportional to the vector $(1/u_i'(u_i^{-1}(v^i_{\{i,j\}})), 1/u_j'(u_j^{-1}(v^j_{\{i,j\}})))$, and also

$$
\begin{aligned}
& (\lambda^1_{\{1,2,3\}}, \lambda^2_{\{1,2,3,\}}, \lambda^3_{\{1,2,3\}}) \\
& \quad = k_{\{1,2,3\}}\left(\frac{1}{u_1'(u_1^{-1}(v^1_{\{1,2,3\}}))}, \frac{1}{u_2'(u_2^{-1}(v^2_{\{1,2,3\}}))}, \frac{1}{u_3'(u_3^{-1}(v^3_{\{1,2,3\}}))}\right)
\end{aligned}
$$

for some scalar $k_{\{1,2,3\}}$.

It suffices to check that the subcoalition-consistent Nash bargaining solution a_S, $S \subset N$, does not satisfy the equations system. It is easy to show that

$$
\begin{aligned}
& \lambda^1_{\{1,2,3\}}(a^1_{\{1,2,3\}} - \frac{1}{2} a^1_{\{1,3\}}) = \frac{k_{\{1,2,3\}}}{6}(3 - \frac{1}{2} 3^{1/2}) \\
& \neq \frac{k_{\{1,2,3\}}}{3} \frac{1}{6^{2/3}}(6^{1/3} - \frac{1}{2} 2^{1/3}) = \lambda^3_{\{1,2,3\}}(a^3_{\{1,2,3\}} - \frac{1}{2} a^3_{\{1,3\}})
\end{aligned}
$$

Thus, our bargaining procedure is not necessarily lead to the subcoalition-consistent Nash bargaining solution as $\rho \to 1$ in a NTU game.

4 Proportionality condition

We will provide a sufficient condition to generate the subcoalition-consistent Nash bargaining solution in the limit of SSPEs of the noncooperative bargaining game $G^N(\rho)$ as $\rho \to 1$. First we add the assumption.

Assumption 2. For all $i, j \in S$, $V(\{i,j\}) \neq \emptyset$ and $(0,0) \notin \partial V(\{i,j\})$.

Next, let us introduce some notations and definitions. Any point in $\mathbb{R}^S$ is denoted by $y_S = (y_S^i)_{i \in S}$. For $i, j \in S$ and $y_S \in \mathbb{R}^S$, $y_S^{-\{i,j\}}$ denotes the $(|S|-2)$-dimensional vector constructed from y_S by deleting the i-th and the j-th coordinates y_S^i, y_S^j in y_S. The point y_S can be written as $(y_S^i, y_S^j, y_S^{-\{i,j\}})$. For any $S \subset N$, $|S| \geq 3$, $y_S \in V(S)$ and $i, j \in S$, we can define the following set:

$$\partial V(S)\big|_{y \in V(S)}^{\{i,j\}} = \left\{ (z_S^i, z_S^j) \in \mathbb{R}_+^2 \,\middle|\, (z_S^i, z_S^j, y_S^{-\{i,j\}}) \in \partial V(S) \text{ and } y_S \in V(S) \right\}.$$

By the notion of $\partial V(S)|_{y \in V(S)}^{\{i,j\}}$, the following condition of the coalitional form game (N, V) can be introduced.

Proportionality condition: For every $S \subset N$ such that $|S| \geq 3$ and for every $i, j \in S$, the set $\partial V(S)|_{y \in V(S)}^{\{i,j\}}$ is represented as a proportional transformation of the set $\partial V(\{i,j\}) \cap \mathbb{R}_+^2$; for any $(z_S^i, z_S^j) \in \partial V(S)|_{y \in V(S)}^{\{i,j\}}$, there exists $k \in \mathbb{R}_+$ and $(y^i, y^j) \in (\partial V(\{i,j\}) \cap \mathbb{R}_+^2)$ such that $(kz_S^i, kz_S^j) = (y^i, y^j)$, where k is common to all points in $\partial V(S)|_{y \in V(S)}^{\{i,j\}}$.

Figure 1 depicts the Proportionality condition. The boundary $\partial V(S)|_{y \in V(S)}^{\{i,j\}}$ is given by a radial expansion of the set $\partial V(\{i,j\})$. In addition, the slopes of a tangent of the boundary set are unchanged along any ray through the origin.

(insert Figure 1 around here. Figure1equalsplit.tex)

It is easy to see that any TU game satisfies the Proportionality condition. Furthermore, the Proportionality condition is also satisfied by a hyperplane coalitional form game (N, V) with the common weight vector λ_N, in which the characteristic function is represented by $V(S) = \{c \in \mathbb{R}^S_+ | \lambda_S \cdot c \leq w_S, w_S \in \mathbb{R}_+\}$ and λ_S is the restriction of λ_N to S.

Consider the division of a pie. As seen in the example before, the corresponding characteristic function V is defined by

$$V(S) = \{v_S \in \mathbb{R}^S_+ \mid \exists (x_i)_{i \in S} \in X_S, \forall i \in S, v_i^S \leq u_i(x_i)\} \text{ for } \forall S \subset N,$$

where X_S is the set of feasible allocation for a coalition S and x^i is the share of the pie to player i. If $u_i(x_i) = x_i$, the coalitional form game becomes a TU game, and if $u_i(x_i) = b^i x_i$, where the value of b^i is not necessarily common among players, the corresponding coalitional game is in a hyperplane coalitional form game with the common weight vector. Moreover, if every player has an identical payoff function, the game is an NTU game, but satisfies the Proportionality condition.

In general, players' preferences are defined on the set of lotteries over X_S, rather than simply on X_S itself (see Osborne and Rubinstein, 1990). The function $u_i(\cdot)$ is interpreted as a von Neumann Morgenstern utility function. The curvature of u_i represents a player's risk attitude. If u_i is concave, player i is risk averse, and if $u_i(x_i) = x_i$, i is risk neutral. Then, the above examples (such as $u_i(x_i) = x_i$; a TU game, $u_i(x_i) = b^i x_i$; a hyperplane coalitional form game with the common weight, and the identical utility function case; $u_i(x_i) = u(x_i)$ for all $i \in N$) represents the same risk attitude of all players. If all players have the same risk attitude, then the corresponding coalitional form game satisfies the Proportionality condition[2]. The relationship and interpretation between the two-person Nash bargaining solution and the risk aversion of players has been examined by Kihlstrom et al. (1981) and Rubinstein et al. (1992).

Theorem 2. *Let (N, V) be a coalitional form game satisfying Assumption 1, 2 and the Proportionality condition. Then, any SSPE payoff vectors in the noncooperative bargaining game $G^N(\rho)$ converge to the subcoalition-consistent Nash bargaining solution of (N, V) as $\rho \to 1$.*

Proof. When the game (N, V) satisfies Assumption 1, the boundary $\partial V(S) \cap \mathbb{R}^S_+$ for each $S \subset N$ is expressed as the set $\{v_S \in \mathbb{R}^S_+ | H^S(v_S) = 0\}$, where $H^S(\cdot)$ is a continuous, concave and differentiable function. Thus, there exists a function H^S such that $H^S(v_S) = 0$ for all $v_S \in \partial V(S) \cap \mathbb{R}^S_+$ and $H^S(v_S) \geq 0$ for all $v_S \in V(S) \cap \mathbb{R}^n_+$. The maximization problem to obtain the Nash bargaining solution of the bargaining problem $(V(S), r_S)$, where $r_S = 0$, is represented by

$$\max_{v_S} \prod_{i \in S} v^i_S \text{ subject to } H^S(v_S) \geq 0, \text{ and } v^i_S \geq 0 \text{ for all } i \in S.$$

We can derive the Kuhn-Tucker condition:

$$\begin{aligned} &\frac{\partial H^S(v^*_S)}{\partial v^{1*}_S} v^{1*}_S = \frac{\partial H^S(v^*_S)}{\partial v^{2*}_S} v^{2*}_S = \cdots = \frac{\partial H^S(v^*_S)}{\partial v^{|S|*}_S} v^{|S|*}_S, \\ &H^S(v^*_S) = 0. \end{aligned}$$

The Nash bargaining solution v^*_S satisfies the Kuhn-Tucker condition. Since the unit supporting normal λ^*_S to $\partial V(S)$ at v^*_S is proportional to the vector $(\partial H^S(v^*_S)/\partial v^1_S, \ldots, \partial H^S(v^*_S)/\partial v^{|S|}_S)$, then the Kuhn-Tucker condition is

[2] We thank the referees who put forward an interpretation from the viewpoint of risk attitudes. The Proportionality condition is inspired by this suggestion.

rewritten as

$$\lambda_S^{1*} v_S^{1*} = \cdots = \lambda_S^{|S|*} v_S^{|S|*},$$
$$H^S(v_S^*) = 0.$$

By the Proportionality condition, there exists a scalar k_S such that $v_S^{i*} = k_S v_{\{i,j\}}^*$ and $(\lambda_S^{i*}, \lambda_S^{j*}) = (\lambda_{ij}^{i*}, \lambda_{ij}^{j*})$ for all $i, j \in N$, where a coalition S contains i, j.

We can easily see that the above subcoalition-consistent Nash bargaining solution satisfies equations (1) in Theorem 1 uniquely. This implies the theorem. □

As seen in the proof of the Theorem 2, we can obtain the Nash bargaining solution of each subcoalition $S \subset N$ if the game follows Assumption 1, 2 and the Proportionality condition. First, we seek the Nash bargaining solution of the bilateral bargaining problem $(V(\{i, j\}), (0, 0))$ for each $i, j \in S$, and we obtain payoff ratios between player i and j for all $i, j \in S$. Next, maintaining the payoff ratio between all players in S, we choose a point in the boundary of $V(S)$. This is the Nash bargaining solution of the bargaining problem $(V(S), r_S)$, where $r_S = 0$.

Definition 2. We say that $V(\{i, j\})$ is *symmetric* if $(v_{\{i,j\}}^i, v_{\{i,j\}}^j) \in V(\{i, j\})$ if and only if $(v_{\{i,j\}}^j, v_{\{i,j\}}^i) \in V(\{i, j\})$.

If $V(\{i, j\})$ is symmetric, the Nash bargaining solution of the 2-person bargaining problem $(V(\{i, j\}), (0, 0))$ is the equal payoff allocation. Then, we have the following corollary.

Corollary 1. *If the coalitional form game (N, V) satisfies Assumption 1, 2 and the Proportionality condition and $V(\{i, j\})$ is symmetric for all $i, j \in N$,* *then any SSPE payoff configurations in $G^N(\rho)$ converge to the "equal split" solution as $\rho \to 1$.*

Thus, the "equal split" solution in Hart and Mas-Colell (1996) is implementable through the noncooperative bargaining game $G^N(\rho)$ in very restricted situations.

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Appendix: Proof of Theorem 1

In order to prove Theorem 1, we prepare the following lemma. This lemma characterizes the proposal for each player in an SSPE and shows that no delay occurs in equilibrium.

Lemma 1. *In every SSPE σ of the game $G^N(\rho)$ with the payoff configuration $\{v_S \mid S \subset N\}$, the corresponding proposal are always accepted, and the equilibrium proposal $a_{S,i} = (a^j_{S,i})_{j\in N}$ in the subgame $G^S(\rho)$ for each $S \subset N$ and each $i \in S$ is characterized by:*

$$a_{S,i} \in \partial V(S), \tag{2}$$

$$a^j_{S,i} = \rho v^j_S + (1-\rho)\sum_{k\in S\backslash i}\frac{1}{|S|-1}v^j_{S\backslash k} \quad \textit{for all } j \in S, i \neq j. \tag{3}$$

Proof. The proof is by induction on the number of players. The lemma holds trivially for the 1-player case. Assume that it holds for the less than n-players case. Let $a_{S,i}$, $i \in S$, be the proposals of player i in an SSPE. Let define the vector $a_S = (1/|S|)\sum_{i\in S} a_{S,i}$. Then, the jth component of a_S is $a^j_S = (1/|S|)\sum_{i\in S} a^j_{S,i}$. We denote the expected payoff vector for the members of S by v^S in the subgames $G^S(\rho)$. By induction hypothesis, $a_S = v_S$.

Since $V(N)$ is convex and v^N is a convex combination of points in $V(N)$, it holds that $v^N \in V(N)$. Since $v_{N\backslash k} \in V(N\backslash k)$ for all $k \in N$, it follows from the monotonicity of V that the vector $(v_{N\backslash k}, 0) = (v^1_{N\backslash k}, \ldots, v^{k-1}_{N\backslash k}, 0, v^{k+1}_{N\backslash k}, \ldots, v^n_{N\backslash k})$ belongs to $V(N)$. Now, we denote by $a^k_{N\backslash k}$ the kth coordinate of $(v_{N\backslash k}, 0)$. Then, $(v_{N\backslash k}, 0)$ is represented by $(v_{N\backslash k}, v^k_{N\backslash k})$. The convexity of $V(N)$ implies that for all $i \in N$

$$\frac{1}{n-1}\sum_{k\in N\backslash i}(v_{N\backslash k}, v^k_{N\backslash k}) = (\frac{1}{n-1}\sum_{k\in N\backslash i} v^1_{N\backslash k}, \ldots, \frac{1}{n-1}\sum_{k\in N\backslash i} v^n_{N\backslash k}) \in V(N).$$

Hence, the convex combination

$$\rho v_N + (1-\rho)\frac{1}{n-1}\sum_{k\in N\backslash i}(v_{N\backslash k}, v^k_{N\backslash k}) \in V(N) \text{ for all } i \in N.$$

If increasing in the ith coordinate of the vector $\rho v_N + (1-\rho)(1/(n-1))\sum_{k\in N\backslash i}(v_{N\backslash k}, v^k_{N\backslash k})$ until reaching the boundary $\partial V(N)$, we can obtain the vector d_i, which satisfies $d^j_i = \rho v^j_N + (1-\rho)(1/(n-1))\sum_{k\in N\backslash i} v^j_{N\backslash k}$ for $j \neq i$ and $d^i_i \geq \rho v^i_N + (1-\rho)(1/(n-1))\sum_{k\in N\backslash i} v^i_{N\backslash k}$. The vector d_i satisfies (2), (3). For $j \neq i$, the amount d^j_i is the expected payoff of j when player j would rejects i's proposal. Therefore, d_i is the best proposal for i among the proposals that will be accepted if i is the proposer. Furthermore, if i makes any proposal that is rejected, then i obtains at most $\rho v^i_N + (1-\rho)(1/(n-1))\sum_{k\in N\backslash i} v^i_{N\backslash k}$, which is less than d_i^i. Hence, player i will propose $a_{N,i} = d_i$ and the proposal will be accepted. Then, we have $v_N = a_N$ by definition of $G^N(\rho)$. $\square$

Let us give the proof of Theorem 1. First, we prove the existence of an SSPE of $G^N(\rho)$ for any $0 \leq \rho < 1$.

(Existence): We proceed by induction. It is trivial for $n = 1$. Assume that the theorem holds for the less than n-players case. This hypothesis implies that there exists an SSPE for every subgame $G^S(\rho)$ for $S \neq N$ and $S \subset N$. We fix one equilibrium strategy combination in $G^S(\rho)$. Let $v_S^*(\rho)$ be the expected payoff vector in $G^S(\rho)$. What we have to do is to construct equilibrium strategies for all remaining nodes in $G^N(\rho)$.

Let us take an element $v_N = (v_N^i)_{i \in N}$ in $V(N)$. We denote by v_N^{-i} the $(n-1)$-dimensional vector constructed from v_N by deleting the ith coordinate v_N^i. Suppose that player i becomes the proposer at round 1 and proposes a solution of the maximization problem:

$$\max_{y_{N,i} \in V(N)} y_{N,i}^i \text{ subject to } y_{N,i}^j \geq \rho v_N^j + \frac{1-\rho}{n-1} \sum_{k \in N \setminus i} v_{N \setminus k}^{j*}(\rho), \text{ for } j \in N, j \neq i.$$

Let $g_{N,i}^i(v_N^{-i})$ be the maximum value which is attained in the maximization problem. By Berge's maximum theorem, $g_{N,i}^i(\cdot)$ is a continuous function with respect to v_N^{-i}.

For any $v_N \in V(N)$, we can define a function

$$\xi_i^\rho(v_N) = \frac{1}{n} g_{N,i}^i(v_N^{-i}) + \frac{1}{n} \sum_{j \in N \setminus i} \Big[\rho v_N^i + \frac{(1-\rho)}{n-1} \sum_{k \in N \setminus j} v_{N \setminus k}^{i*}(\rho)\Big].$$

Moreover, let define $\xi^\rho(v_N) = \prod_{i \in N} \xi_i^\rho(v_N)$. The function $\xi^\rho(v_N)$ is a continuous function from $V(N)$ to itself. In addition, the set $V(N)$ is compact and convex. Then, by Brouwer's fixed point theorem, there exists a fixed point $v_N^*(\rho) = (v_N^{1*}(\rho), \ldots, v_N^{n*}(\rho)) \in V(N)$ such that for all $i \in N$,

$$v_N^{i*}(\rho) = \frac{1}{n} g_{N,i}^i(v_N^{-i*}(\rho)) + \frac{1}{n} \sum_{j \in N \setminus i} \Big[\rho v_N^{i*}(\rho) + \frac{(1-\rho)}{n-1} \sum_{k \in N \setminus j} v_{N \setminus k}^{i*}(\rho)\Big].$$

Using the fixed point $v_N^*(\rho)$, we can construct an SSPE of $G^N(\rho)$. Consider the strategy combination σ^* such that

(a) every player i as a proposer offers a solution of the maximization problem:

$$\max_{y_{N,i} \in V(N)} y_{N,i}^i \text{ subject to } y_{N,i}^j \geq \rho v_N^{j*}(\rho) + \frac{1-\rho}{n-1} \sum_{k \in N \setminus i} v_{N \setminus k}^{j*}(\rho), \text{ for } j \in N, j \neq i.$$

In other words, player i proposes

$$a_{N,i} \in \partial V(N), \text{ and}$$
$$a_{N,i}^j = \rho v_N^{j*}(\rho) + (1-\rho) \sum_{k \in N \setminus k} \frac{1}{n-1} v_{N\setminus k}^{j*}(\rho), \text{ for } j \in N, j \neq i.$$

(b) player i as a responder accepts any proposal y^i in the case that player j is a proposer if and only if

$$y^i \geq \rho v_N^{i*}(\rho) + (1-\rho) \sum_{k \in N \setminus j} \frac{1}{n-1} v_{N\setminus k}^{i*}(\rho).$$

By Lemma 1, it is easy to check that σ^* prescribes every player's (locally) optimal choice at his every move in $G^N(\rho)$. Thus, σ^* is an SSPE of $G^N(\rho)$ with the payoff configuration $\{v_S^*(\rho) | S \subset N\}$.

(SSPE payoff configuration): Next, we introduce the notion of a *hyperplane coalitional form game* $(N, \tilde{V})$. In this game, each $\tilde{V}(S)$ is defined as a half space in $\mathbb{R}_+^S$. Thus, the set $\tilde{V}(S)$ is represented by for some $\lambda_S \in \mathbb{R}_{++}^S$,

$$\tilde{V}(S) \stackrel{\text{def}}{=} \left\{ c \in \mathbb{R}^S \;\middle|\; \sum_{i \in S} \lambda_S^i c^i \leq w_S \right\}.$$

We can get the following lemma.

Lemma 2. *Let $(N, \tilde{V})$ be a hyperplane coalitional form. Then for each $0 \leq \rho < 1$ there exist a unique SSPE of $G^N(\rho)$. Moreover, the SSPE payoff configuration $(v_S)_{S \subset N}$ satisfies that for all $i \in S \subset N$,*

$$\left[\frac{1}{|S|} \sum_{j \in S} \lambda_S^j v_S^j - \lambda_S^i v_S^i \right] + \sum_{k \in S \setminus i} \frac{1}{|S|} \left[\lambda_S^i v_{S\setminus k}^i - \frac{1}{|S|-1} \sum_{j \in S \setminus k} \lambda_S^j v_{S\setminus k}^j \right] = 0. \qquad (4)$$

Proof. We have already proved the existence of an SSPE of $G^N(\rho)$ in the general NTU game (N, V). Then, we focus on the SSPE payoff configuration in a hyperplane coalitional game. By Lemma 1, we can regard a_S (the average of proposals for each subgame in an SSPE) in the same light as v_S (the expected payoff vector in an SSPE). Thus, $a_S = v_S$ for all $S \subset N$. Here, v_S and a_S is often used interchangeably.

We proceed by induction. Assume the statement is correct for the less than n-players case. Let $\lambda_N \in \mathbb{R}_{++}^n$ and

$$\tilde{V}(N) \stackrel{\text{def}}{=} \left\{ c \in \mathbb{R}^n \;\middle|\; \sum_{i \in N} \lambda_N^i c^i \leq w_N \right\}.$$

By definition of a_N^i and by Lemma 1, it holds that for every $i \in N$

$$\begin{aligned}
n\lambda_N^i v_N^i &= \lambda_N^i a_{N,i}^i + \sum_{j \in N\setminus i} \lambda_N^i a_{N,j}^i \\
&= (w_N - \sum_{j \in N\setminus i} \lambda_N^j a_{N,i}^j) + \sum_{j \in N\setminus i} \lambda_N^i a_{N,j}^i \\
&= w_N - \sum_{j \in N\setminus i} \lambda_N^j (\rho v_N^j + (1-\rho) \sum_{k \in N\setminus i} \frac{v_{N\setminus k}^j}{n-1}) \\
&\qquad + \sum_{j \in N\setminus i} \lambda_N^i (\rho v_N^i + (1-\rho) \sum_{k \in N\setminus j} \frac{v_{N\setminus k}^i}{n-1}).
\end{aligned}$$

Since $w_N = \sum_{j \in N} \lambda_N^j v_N^j$, the above equality is rewritten by

$$\begin{aligned}
n\lambda_N^i v_N^i = (1-\rho) \sum_{j \in N} \lambda_N^j v_N^j + \lambda_N^i \rho v_N^i - \frac{1-\rho}{n-1} \sum_{k \in N\setminus i} \sum_{j \in N\setminus i} \lambda_N^j v_{N\setminus k}^j \\
+ (n-1)\lambda_N^i \rho v_N^i + \frac{1-\rho}{n-1} \sum_{j \in N\setminus i} \sum_{k \in N\setminus k} \lambda_N^i v_{N\setminus k}^i.
\end{aligned}$$

Then, it reduces to the following equality:

$$\begin{aligned}
n(1-\rho)\lambda_N^i v_N^i = (1-\rho) \sum_{k \in N\setminus i} \lambda_N^i v_{N\setminus k}^i + (1-\rho) \sum_{j \in N} \lambda_N^j v_N^j \\
- \frac{1-\rho}{n-1} \sum_{k \in N\setminus i} \sum_{j \in N\setminus i} \lambda_N^j v_{N\setminus k}^j.
\end{aligned}$$

By dividing the last equality by $n(1-\rho)$ and rearranging, we obtain equation (4). □

By Lemma 2, we can derive the SSPE payoff configuration in Theorem 1 as $\rho \to 1$.

Let $\lambda_S(\rho)$ be the outward unit normal to the hyperplane passing through the vector $\{a_{S,i} \mid i \in S\}$, and let $\tilde{V}_\rho(S)$ be the half-space below the hyperplane. Then, we have a hyperplane coalitional form game $(N, \tilde{V}_\rho)$ for each ρ.

By Lemma 1, we have, for all $S \subset N$ and for all $i \in S$,

$$\lim_{\rho \to 1} a_{S,i} = \lim_{\rho \to 1} v_S^*(\rho) = a_S = v_S. \tag{5}$$

Furthermore, it follows from Assumption 1 that the boundary $\partial V(S) \cap \mathbb{R}_+^{|S|}$ is smooth and nonlevel. Hence, we have $\lambda_S(\rho) \to \lambda_S$ as $\rho \to 1$. Therefore, we obtain

$$\tilde{V}_\rho(S) \to \tilde{V}'(S) \stackrel{\text{def}}{=} \{c \in \mathbb{R}^S \mid \lambda_S c \leq \lambda_S v_S\}.$$

It is clear from Lemma 1 that the payoff configuration $v(\rho)$ remains an SSPE payoff configuration for the hyperplane coalitional form $(N, \tilde{V}_\rho)$. By Lemma 2, $v(\rho)$ satisfies the equation (4) of $(N, \tilde{V}_\rho)$. Therefore, as $\rho \to 1$, the limit point $(v_S)_{S \subset N}$ of the SSPE payoff configuration of the game (N, V) is precisely the payoff configuration of the hyperplane form $(N, \tilde{V}')$, which satisfies equation (4). Note that equation (1) is in the same form as equation (4). This completes the proof of Theorem 1.

Figure 1