Inter-generational and Intra-generational Redistribution and Stability of the Golden Rule Path

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Abstract

This paper examines the effects of intra-generational and inter-generational redistribution on the stability of the golden rule path in a consumption-loan economy. Every individual lives for three periods. If the rate of time preference for a low-income individual is greater than that for a high-income individual, correcting the intra-generational inequality between persons in the second period of their lives increases the stability of the golden rule path in the consumption-loan economy and vise versa. Moreover, the correction of inter-generational inequality between a person in the first period and a person in the second period always decreases the stability of the golden rule steady state equilibrium.

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Key Words: Income Redistribution, Stability, Golden-Rule Path.

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1 Introduction

This paper examines effects of correcting the income differentials in a certain period on the stability of the golden rule steady state in the consumption-loan model. The consumption-loan model, which was provided by Samuelson (1958), is a simple dynamic model of exchange economy with overlapping-generations. It is well known that the golden-rule steady-state equilibrium in which the interest rate exactly equals the population growth rate is Pareto optimal in the consumption-loan economy. Samuelson showed that there exists a golden rule steady state in every consumption-loan economy, but that such a steady state can never be achieved dynamically by the consumption-loan market equilibrium. This result has been known as the impossibility theorem of welfare economics in a dynamic competitive economy. Gale (1973) perfectly characterized the global stability of the golden rule steady state in the consumption-loan model in which each person lives for two periods. Homma (1977) then explicitly presented a stability condition of the golden rule path for the three-period model as in Samuelson. These studies assumed that all people are identical as to income streams and preference. We extend the model with identical people to that with two types of people in each generation. Two types of people living exactly three periods receive different sequences of income in their lifetimes. One is endowed with income of M_0 in the first period of his life, M_0 in the second period, and 0 in the last period, i.e., is endowed with income sequences $\{M_0, M_0, 0\}$. The other receives income sequences $\{M_0, (1+m)M_0, 0\}$ for three periods, where $m > 0$. We say that the former is *individual L* and that the latter is *individual H*. This extension of the model will enable us to deal with the inter-generational and intra-generational inequality of income for each period. In this paper, we deal with differences of income between persons in the second period of their lives as the intra-generational inequality and with income differences between a person in the first period and a person in the second period of his life as the inter-generational inequality. Both differences are $(1+m)M_0 - M_0$, and coexist within the same period. Even in the extended model, the golden rule equilibrium path exists as one of the steady state equilibria and the equality between a long-run rate of interest and the growth rate of population is a necessary condition for the biological optimum to be achieved through the consumption-loan market.

This paper investigates the effects of reducing the inter-generational and intra-generational inequalities of income on the stability of the golden rule path in the consumption-loan economy. We obtain the following results. First, the effect of corrections for the intra-generational inequality depends on the rates of time preference for two types of people; individual H and individual L. If the rate of time preference for individual L is greater than that for individual H, the correction of the intra-generational inequality increases the stability of the golden rule path in the consumption-loan economy. Conversely, if the rate of time preference for individual L is less than that for individual H, correcting the intra-generational inequality of income decreases the stability of the golden rule path. Second, the correction of inter-generational inequality always reduces the stability of the golden rule steady state equilibrium. This implies that correcting the inequality of income for a certain period may decrease the long-run social welfare level owing to the economy's deviation from the golden rule path. In addition, these results indicate the possibility that the correction of intra-generational inequality would have opposite effects to that of inter-generational inequality on the stability of the golden rule steady state.

This paper is organized as follows. Section 2 constructs the consumption-loan model in which each person lives for three periods. Section 3 investigates the effect of intra-generational redistribution on the stability of the golden rule path in the consumption-loan economy. Section 4 examines the inter-generational redistribution. The Appendix supplies the proof of Lemma 2.

2 The Three-Period Model

We consider an exact consumption-loan model of interest, as provided by Samuelson (1958) and Homma (1977). Each people lives for three periods and receives income in the early two periods of his or her life but none at the end. People of each generation are divided into two classes. Every person who belongs to the first class receives the income M_0 in the first period of his or her life, $(1+m)M_0$ in the second period, and zero in the third period, where $m > 0$ is the income growth rate. We call a person in the first class *individual* H. A person in the second class receives the same income M_0 in the first and second period of his or her life. We call a person in the second class *individual* L. The population of the first class and that of the second class are equal, and grow at the same rate n. We denote a population of the first class born in period t by L_{Ht} and that of the second class born in period t by L_{Lt}.

Let $C_H(t) = (C_{H0}(t), C_{H1}(t+1), C_{H2}(t+2))$ and $C_L(t) = (C_{L0}(t), C_{L1}(t+1), C_{L2}(t+2))$ be the consumption in the three periods of the life of individual H and that of individual L born in period t. The utility functions of individual H and individual L are given by the Cobb-Douglas fashion:

$$U_H(C_H(t)) \stackrel{\text{def}}{=} \sum_{j=0}^{2} (1+\delta_H)^{-j} \log C_{Hj}(t+j), \tag{1}$$

$$U_L(C_L(t)) \stackrel{\text{def}}{=} \sum_{j=0}^{2} (1+\delta_L)^{-j} \log C_{Lj}(t+j), \tag{2}$$

where δ_H (δ_L) is a pure rate of time preference for individual H (for individual L).

Each individual chooses the sequence of his or her lifetime consumption to maximize utility (1) or (2) subject to individual's budget constraints. We define the *net savings* of each person by the algebraic differences between his or her consumption and income for one period. Note that net savings become negative when the individual's consumption exceeds his or her income. The equilibrium condition of the consumption-loan market is that aggregate net saving for the economy canceled out to zero for every period. We denote the interest rate of the consumption-loan market in period t by $\rho(t)$. The present value factor $R(t)$ is defined by $R(t) = 1/(1+\rho(t))$, which represents the discount rate of the consumption good of period t for the consumption good of period $t+1$.

A steady state equilibrium in which the interest rate equals the biological rate of population is called the golden rule path. The golden rule path is a Pareto-optimal allocation in the steady state for the economy, as is shown by Samuelson (1958) and Gale (1973). In other words, the golden rule path coincides with the solution of the representative person's utility maximization problem with a steady-state feasible constraint, the so-called *biological optimum*. Even in our model, a long-run equilibrium rate of interest must equal the population's growth rate in order to achieve the solution of the Benthamite social welfare maximization problem in a steady state equilibrium. Thus, the golden rule, $\rho(t) = n$, is a necessary condition for a steady state competitive equilibrium to implement some kind of a social optimum.

Let focus on any period t in our model. Income differentials exists between individual H and individual L in the second period of their lives. Individual H is endowed with income $(1+m)M_0$ and individual L is endowed with income M_0 in the second period. We call the dispersion of income between persons of the same generation *intra-generational inequality*, and call the redistribution of income in order to reduce the intra-generational inequality *intra-generational redistribution*. Moreover, income differential exists between persons of different generations. Individual H in the first period of his life has an income of M_0 and individual H in the second period has an income of $(1+m)M_0$ at the same period t. In addition, a person in the third period of his or her life obtains no income, but a person in the first or second period obtains an income of M_0 or $(1+m)M_0$. We call the dispersion of income between persons of different generations *intergenerational inequality*. Similarly, the redistribution in order to correct intergenerational inequality is called *inter-generational redistribution*.

We consider the effects of intra-generational and inter-generational redistribution on the stability of the golden rule path. We restrict inter-generational redistribution to the redistribution between persons with income M_0 in the first period and individual H with income $(1+m)M_0$ in the second period of his or her life, and do not examine the effects of redistribution between persons in the third period and others. There are two reasons for this restriction. First, intra-generational inequality and inter-generational inequality become the same size because intra-generational inequality is the difference between income M_0 of individual L and income $(1+m)M_0$ of individual L in the second period of his or her life; thus, $(1+m)M_0 - M_0$. The second reason is a technical one. We can exclude the *no-trade equilibrium* as in Samuelson (1958) and Gale (1973) by restricting the redistribution and leaving persons in the third period without income.

3 Intra-generational Redistribution

Let us examine the effect of the intra-generational redistribution on the dynamic stability of the golden rule path. Concretely, the redistribution is conducted by means of income taxation in which the government, at each period, levies a tax on income $(1+m)M_0$ of individual H in the second period of his or her life with an average tax rate τ and transfers the tax revenue T to individual L in the second period of his or her life. Individual H and individual L born in period t each face the following lifetime budget constraints:

$$C_{H0}(t) + R(t)C_{H1}(t+1) + R(t+1)R(t)C_{H2}(t+2)$$
$$= M_0[1 + (1+m)(1-\tau)R(t)], \tag{3}$$
$$C_{L0}(t) + R(t)C_{L1}(t+1) + R(t+1)R(t)C_{L2}(t+2)$$
$$= M_0[1 + R(t)] + TR(t), \tag{4}$$

where M_0 is the income received in the first period of his or her life, and where $R(t)$ is the present value factor in period t.

The government budget equation for each period is given by

$$T = \tau(1+m)M_0. \tag{5}$$

We limit the range of tax rate τ so as not to reverse the order of income groups; individual H and individual L in the second period of their lives. Thus, the government sets a tax rate less than the rate at which the incomes of individual H and individual L in the second period are equalized. Formally, an income tax rate τ is assumed to satisfy the following inequality,

$$0 \leq \tau \leq \frac{m}{2(1+m)}. \tag{6}$$

Each person maximizes (1) or (2) subject to (3) or (4). Then, the following consumption functions of individual H and individual L born in period t are given by

$$C_{H0}(t) = \frac{1+(1+m)(1-\tau)R(t)}{1+\beta_H+\beta_H^2}M_0, \tag{7}$$

$$C_{H1}(t+1) = \frac{\beta_H}{1+\beta_H+\beta_H^2}\frac{1+(1+m)(1-\tau)R(t)}{R(t)}M_0, \tag{8}$$

$$C_{H2}(t+2) = \frac{\beta_H^2}{1+\beta_H+\beta_H^2}\frac{1+(1+m)(1-\tau)R(t)}{R(t)R(t+1)}M_0, \tag{9}$$

$$C_{L0}(t) = \frac{1+(1+\tau(1+m))R(t)}{1+\beta_L+\beta_L^2}M_0, \tag{10}$$

$$C_{L1}(t+1) = \frac{\beta_L}{1+\beta_L+\beta_L^2}\frac{1+(1+\tau(1+m))R(t)}{R(t)}M_0, \tag{11}$$

$$C_{L2}(t+2) = \frac{\beta_L^2}{1+\beta_L+\beta_L^2}\frac{1+(1+\tau(1+m))R(t)}{R(t)R(t+1)}M_0, \tag{12}$$

where $\beta_H = 1/(1+\delta_H)$ ($\beta_L = 1/(1+\delta_L)$) is the discount factor of individual H (the discount factor of individual L).

In equilibrium, aggregate net savings must be equal to zero. That is, the equilibrium condition for each t is

$$\begin{aligned}(M_0 - C_{H0}(t))L_{Ht} + (M_0 - C_{L0}(t))L_{Lt} \\ + ((1-\tau)(1+m)M_0 - C_{H1}(t))L_{Ht-1} + (M_0 + T - C_{L1}(t))L_{Lt-1} \\ + (0 - C_{H2}(t))L_{Ht-2} + (0 - C_{L2}(t))L_{Lt-2} = 0.\end{aligned} \tag{13}$$

By substituting (7)-(12) to (13), we obtain the nonlinear difference equation

$$R(t) = a_1 + \frac{a_2}{R(t-1)} + \frac{a_3}{R(t-1)R(t-2)}, \tag{14}$$

where

$$
\begin{aligned}
a_1 \overset{\text{def}}{=} \Big[&(\beta_H + \beta_H^2)(1 + \beta_L + \beta_L^2) + (\beta_L + \beta_L^2)(1 + \beta_H + \beta_H^2) \\
&(x(1+m)(1-\tau) + x(1+\tau(1+m)))\,(1 + \beta_H + \beta_H^2)(1 + \beta_L + \beta_L^2) \\
&- \beta_H x(1 + \beta_L + \beta_L^2)(1+m)(1-\tau) \\
&\quad - \beta_L x(1 + \beta_H + \beta_H^2)(1 + \tau(1+m))\Big] \\
&/ \left((1 + \beta_L + \beta_L^2)(1+m)(1-\tau) + (1 + \beta_H + \beta_H^2)(1 + \tau(1+m))\right), \\
a_2 \overset{\text{def}}{=} \Big[&\beta_H x(1 + \beta_L + \beta_L^2) + \beta_H^2 x^2(1 + \beta_L + \beta_L^2)(1+m)(1-\tau) \\
&+ \beta_L x(1 + \beta_H + \beta_H^2) + \beta_L^2 x^2(1 + \beta_H + \beta_H^2)(1 + \tau(1+m))\Big] \\
&/ \left((1 + \beta_L + \beta_L^2)(1+m)(1-\tau) + (1 + \beta_H + \beta_H^2)(1 + \tau(1+m))\right), \\
a_3 \overset{\text{def}}{=} &- \left(\beta_H^2 x^2(1 + \beta_L + \beta_L^2) + \beta_L^2 x^2(1 + \beta_H + \beta_H^2)\right) \\
&/ \left((1 + \beta_L + \beta_L^2)(1+m)(1-\tau) + (1 + \beta_H + \beta_H^2)(1 + \tau(1+m))\right),
\end{aligned}
$$

and $x = 1/(1+n)$.

Since $\lambda = R(*) = R(t) = R(t-1) = R(t-2)$ in the steady state, the following characteristic equation associated with (14) can be derived from the equilibrium condition;

$$\frac{1}{\lambda^2}(\lambda - x)(b_1\lambda^2 - b_2\lambda - b_3) = 0, \tag{15}$$

where

$$
\begin{aligned}
b_1 &\stackrel{\text{def}}{=} (1+\beta_L+\beta_L^2)(1+m)(1-\tau)+(1+\beta_H+\beta_H^2)(1+\tau(1+m)),\\
b_2 &\stackrel{\text{def}}{=} (\beta_H+\beta_H^2)(1+\beta_L+\beta_L^2)+(\beta_L+\beta_L^2)(1+\beta_H+\beta_H^2)\\
&\quad +x((1+m)(1-\tau)+(1+\tau(1+m)))(1+\beta_H+\beta_H^2)(1+\beta_L+\beta_L^2)\\
&\quad -\beta_H x(1+\beta_L+\beta_L^2)(1+m)(1-\tau)-\beta_L x(1+\beta_H+\beta_H^2)(1+\tau(1+m))\\
&\quad -x(1+\beta_L+\beta_L^2)(1+m)(1-\tau)-x(1+\beta_H+\beta_H^2)(1+\tau(1+m)),\\
b_3 &\stackrel{\text{def}}{=} \beta_H^2 x(1+\beta_L+\beta_L^2)+\beta_L^2 x(1+\beta_H+\beta_H^2).
\end{aligned}
$$

Equation (15) is a cubic equation with respect to λ. There are then at most three possible steady state equilibria. It is clear from (15) that $\lambda = x$ is one of the steady state equilibria. Since

$$
\lambda = R(*) = \frac{1}{1+\rho(*)} = \frac{1}{1+n} = x,
$$

the following proposition can be obtained:

Proposition 1. *The golden rule steady state equilibrium exists as one of the steady state equilibria in the consumption-loan economy.*

Moreover, with regard to the characteristic roots of (15), we can prove a useful property of the dynamic motion of the present value factor in consumption-loan market equilibrium. The following lemma is mainly based to Homma (1977).

Lemma 2. *The equilibrium time path of the present value factor $R(t)$ asymptotically approaches the steady state present value factor corresponding to the characteristic root which has the largest modulus as time tends to infinity.*

Proof. See Appendix. □

Three characteristic roots of (15) are explicitly derived as follows:

$$\lambda_1 = x, \quad \lambda_2 = \frac{b_2 + \sqrt{(b_2)^2 + 4b_1 b_3}}{2b_1}, \quad \lambda_3 = \frac{b_2 - \sqrt{(b_2)^2 + 4b_1 b_3}}{2b_1}. \tag{16}$$

According to Lemma 2, root $\lambda_1 = x$ must have the largest modulus in order for the consumption-loan market equilibrium to converge to the golden rule steady state. It always holds that $\lambda_2 > \lambda_3$ under $b_1, b_2, b_3 > 0$. Therefore, we satisfy the condition $\lambda_1 > \lambda_2$ in order for the golden rule steady state to be stable along the time path of the consumption-loan equilibrium. Then,

$$x > \frac{b_2 + \sqrt{(b_2)^2 + 4b_1 b_3}}{2b_1}. \tag{17}$$

By the definition of b_1, b_2, b_3 and rearranging (17), we can obtain the stability condition

$$\begin{aligned} \psi(\tau) \stackrel{\text{def}}{=} & (1 + \beta_L + \beta_L^2)x \left[x(1+m)(1-\tau)(1-\beta_H^2) - \beta_H(1+2\beta_H)\right] \\ & + (1 + \beta_H + \beta_H^2)x \left[x(1+\tau(1+m))(1-\beta_L^2) - \beta_L(1+2\beta_L)\right] > 0. \end{aligned} \tag{18}$$

If above condition is satisfied, the consumption-loan market economy converges to the golden rule steady state.

We examine whether the redistribution increases the stability of the golden rule path, concretely, extends the range of parameters so as to achieve the golden rule steady state. For this purpose, it is sufficient to check whether or not the right side $\psi(\tau)$ of (18) is increased by introducing income tax τ. By differentiating $\psi(\tau)$ with respect to tax rate τ, we find

$$\frac{d\psi(\tau)}{d\tau} = (1+m)x^2 \left[(2\beta_H + 2\beta_L + \beta_H\beta_L)(\beta_H - \beta_L)\right]. \tag{19}$$

Because $m, x, \beta_H, \beta_L > 0$, $d\psi(\tau)/\tau$ is positive (negative) if $\beta_H > \beta_L$ ($\beta_H < \beta_L$). We then have the following propositions:

Proposition 3. *If the discount factor β_H of individual H is larger than the β_L of individual L; $\beta_H > \beta_L$, the (intra-generational) redistribution between individual H and individual L in the second period of his or her life increases the stability of the golden rule path in the consumption-loan economy.*

Proposition 4. *If the discount factor β_H of individual H is less than the β_L of individual L; $\beta_H < \beta_L$, the (intra-generational) redistribution between individual H and individual L in the second period of his or her life reduces the stability of the golden rule path in the consumption-loan economy.*

4 Inter-generational Redistribution

Next, we examine the effect of inter-generational redistribution on the stability of the golden rule path. In order to extract the pure effect of the inter-generational redistribution and simplify our mathematical calculations, we remove the intra-generational inequality from the model. Thus, we consider the model with only the previous section's individual H, who is endowed with income M_0 in the first period, $(1+m)M_0$ in the second period, and zero in the third period of his or her life. Here, all people are identical. Then, we will abbreviate the subscripts H and L of each variable.

Even in this model, there exists inter-generational inequality. Inter-generational redistribution is conducted by collecting income taxes θ from persons in the second period and transferring the revenue T per person in the first period of his or her life for each period t. The lifetime budget constraint for a person born in period t is given by

$$\begin{aligned} & C_0(t) + R(t)C_1(t+1) + R(t+1)R(t)C_2(t+2) \\ & \quad = M_0 + T + (1+m)(1-\theta)R(t)M_0. \end{aligned} \tag{20}$$

The budget equation for the government in period t is given by

$$TL_t = (1+m)\theta M_0 L_{t-1},$$

where L_t is a population of persons born in period t, and $L_t = (1+n)^t L_0$, $L_{t-1} = (1+n)^{t-1} L_0$. We then obtain

$$T = \frac{1+m}{1+n}\theta M_0. \tag{21}$$

Moreover, we restrict the range of tax rate θ in order to avoid the income reversal as in the previous section. That condition is

$$0 \leq \theta \leq \frac{(1+n)m}{(1+m)(2+n)}.$$

The consumption function of a person born in period t is given by

$$C_0(t) = \frac{1 + (1+m)\theta/(1+n) + (1+m)(1-\theta)R(t)}{1+\beta+\beta^2} M_0, \tag{22}$$

$$C_1(t+1) = \frac{\beta}{1+\beta+\beta^2} \frac{1 + (1+m)\theta/(1+n) + (1+m)(1-\theta)R(t)}{R(t)} M_0, \tag{23}$$

$$C_2(t+2) = \frac{\beta^2}{1+\beta+\beta^2} \frac{1 + (1+m)\theta/(1+n) + (1+m)(1-\theta)R(t)}{R(t+1)R(t)} M_0. \tag{24}$$

The equilibrium condition for each period t is represented by

$$\begin{aligned} & (M_0 + T - C_0(t))L_t + ((1-\theta)(1+m)M_0 - C_1(t))L_{t-1} \\ & \qquad + (0 - C_2(t))L_{t-2} = 0. \end{aligned} \tag{25}$$

Similarly in Section 3, by substituting (21), (22), (23), and (24) to (25) and by rearranging, we are led to the nonlinear difference equation, which describes the dynamic motions of the present value factor in the consumption-loan economy. Then, the characteristic equation associated with the difference equation can be derived by putting $\lambda = R(*) = R(t) = R(t-1) = R(t-2)$ as follows:

$$\frac{1}{\lambda'^2}(\lambda' - x)(b'_1\lambda'^2 - b'_2\lambda' - b'_3) = 0, \tag{26}$$

where

$$\begin{aligned}
b'_1 &\stackrel{\text{def}}{=} (1+m)(1-\theta), \\
b'_2 &\stackrel{\text{def}}{=} \beta\left[(1+\beta)(1+\frac{1+m}{1+n}\theta) + \beta x(1+m)(1-\theta)\right], \\
b'_3 &\stackrel{\text{def}}{=} \beta^2 x(1+\frac{1+m}{1+n}\theta).
\end{aligned}$$

We can easily verify from (26) that the golden rule steady state equilibrium also exists as one of the steady state equilibria in this model. Using Lemma 2 we find the condition for the consumption-loan market equilibrium to converge asymptotically to the golden rule steady state. That is,

$$x > \frac{b'_2 + \sqrt{(b'_2)^2 + 4b'_1b'_3}}{2b'_1}. \tag{27}$$

By the definition of b'_1, b'_2, b'_3 and by rearranging (27), we can obtain the stability condition

$$\phi(\theta) \stackrel{\text{def}}{=} x^2(1+m)(1-\theta)(1-\beta^2) - \beta x(1+2\beta)(1+\frac{1+m}{1+n}\theta) > 0. \tag{28}$$

Condition (28) is the same one derived by Homma (1977) in his *possibility theorem*.

Our concern is whether or not the inter-generational redistribution extends the range of parameters so as to converge globally to the golden rule steady state. Differentiating the left side $\phi(\theta)$ of (28) with respect to tax rate θ, then we can derive

$$\frac{d\phi(\theta)}{d\theta} = -x^2(1+m)(1+\beta+\beta^2) < 0,$$

where $d\phi(\theta)/d\theta < 0$ because $x^2, m, \beta > 0$. Therefore, we have:

Proposition 5. *The inter-generational redistribution between persons in the first period and persons in the second period of their lives reduces the stability of the golden rule path in the consumption-loan economy.*

Appendix: Proof of Lemma 2

The nonlinearity in $R(t)$ of the difference equation (14) makes an analysis difficult to investigate dynamic stability of the consumption-loan market equilibrium. Let us introduce the concept of the total present value factor, defined by $Y(t) = \prod_{\nu=0}^{\nu=t} R(\nu)$, as in Homma (1977). Multiplying both sides of (14) by $Y(t-1)$ leads us to a linear difference equation of the total value factors as follows:

$$Y(t) = a_1 Y(t-1) + a_2 Y(t-2) + a_3 Y(t-3), \tag{A-1}$$

where a_1, a_2, a_3 is as defined in Section 3. Let us define $Z(t) = [Y(t), Y(t-1), Y(t-2)]^T$. Then, (A-1) is expressed as

$$Z(t) = AZ(t-1), \tag{A-2}$$

where

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of matrix A exactly coincide with the characteristic roots of (15). Using the characteristic roots $\lambda_j (j = 1, 2, 3)$ of (15), we can obtain the following solution to the difference equation (A-2):

$$Z(t) = \begin{bmatrix} Y(t) \\ Y(t-1) \\ Y(t-2) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{3} k_j(\lambda_j)^t \\ \sum_{j=1}^{3} k_j(\lambda_j)^{t-1} \\ \sum_{j=1}^{3} k_j(\lambda_j)^{t-2} \end{bmatrix}, \qquad \text{(A-3)}$$

where

$$k_1 = \frac{Y(2) - (\lambda_2 + \lambda_3)Y(1) + \lambda_2\lambda_3 Y(0)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)},$$
$$k_2 = \frac{Y(2) - (\lambda_1 + \lambda_3)Y(1) + \lambda_1\lambda_3 Y(0)}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)},$$
$$k_3 = \frac{Y(2) - (\lambda_1 + \lambda_2)Y(1) + \lambda_1\lambda_2 Y(0)}{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)}.$$

Note that $R(t) = Y(t)/Y(t-1)$. Then, we derive from (A-3) that

$$R(t) = \frac{\sum_{j=1}^{3} k_j(\lambda_j)^t}{\sum_{j=1}^{3} k_j(\lambda_j)^{t-1}}. \qquad \text{(A-4)}$$

We denote the characteristic root of (15) having the largest modulus by λ_m and the coefficient associated with λ_m by k_m. We rewrite (A-4) to

$$R(t) = \frac{\lambda_m + \sum_{j \neq m}(k_j\lambda_j/k_m)(\lambda_j/\lambda_m)^t}{1 + \sum_{j \neq m}(k_j/k_m)(\lambda_j/\lambda_m)^{t-1}}. \qquad \text{(A-5)}$$

Then, $R(t)$ will asymptotically approach λ_m because the second terms of both the numerator and the denominator of (A-5) converge to zero as time t tends to infinity. □

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