PATTERN MATCHING PROBABILITIES AND PARADOXES AS A NEW VARIATION ON PENNEY'S COIN GAME

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Abstract: This paper gives an outline of an interesting probability game related to coin arrangements discovered by Walter Penney in 1969, and explains John Conway's concept of leading numbers and his algorithm for using them to automatically calculate mutual odds of winning. I will show that Conway's elegant algorithm is correct by using average waiting times as proposed by Stanley Collings, and discuss the effect of applying the rules of this coin game to a card game instead.

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1. Penney Ante

One day I received an email from a British acquaintance, Steve Humble. In some excitement, he told me that a magician named Derren Brown was introducing an interesting game on the television. I was a little dubious upon hearing the word " magician ", but after close examination I realized that the game had a mathematical background and was an interesting exercise in probability. The game is not a well known one, so I would like to introduce it to you.

Assume a coin flip with equal probability of coming up heads or tails. The game is played by Players A and B each selecting a sequence of three flips, flipping the coin repeatedly, and seeing whose sequence comes up first. For example, assume that Player A selected " heads-heads-heads " (HHH) and Player B selected " tails-heads-heads " (THH). Then, assume that the coin is flipped repeatedly, resulting in a sequence like the following:

$$\text{HTHTHHHHTHHHTTTTHTHH}\cdots$$

The player whose sequence showed up first, HHH or THH, would be declared the winner.

Both heads and tails have an equal probability of appearing. There are $2^3 = 8$ possible sequences of three flips, and so each sequence has a 1 in 8 chance of appearing. It would seem that both players have an equal chance of winning, regardless of the sequence chosen, but would you believe that the player who selected THH is seven times more likely to win than the player who selected HHH?

2. Non-transitive relations

This problem in probability has been around for some time, but is not widely known. Walter Penney first presented it in an article, only ten lines long, in the *Journal of Recreational Mathematics* in 1969 [1]. Martin Gardner later provided a more detailed description in his " Mathematical Games " column in the October 1974 issue of *Scientific American* [2].

When the game is played using patterns of length 3, no matter what sequence Player A chooses, Player B can always make a winning selection. Table 1 shows the eight possible selections for Player A and the winning selection for Player B. According to the table, selecting THH in response to HHH gives 7:1 odds of winning, selecting THH in response to HHT gives 3:1 odds, and selecting HHT in response to HTH gives 2:1 odds.

A's choice	B's choice	Odds in favor of B
HHH	THH	7 to 1
HHT	THH	3 to 1
HTH	HHT	2 to 1
HTT	HHT	2 to 1
THH	TTH	2 to 1
THT	TTH	2 to 1
TTH	HTT	3 to 1
TTT	HTT	7 to 1

Table 1. Odds in favor of B.

Table 1 shows that THH is stronger than HHH, TTH is stronger than THH, HTT is stronger than TTH, HHT is stronger than HTT, and THH is stronger than HHT. In other words, Player A does not have a strongest selection. This

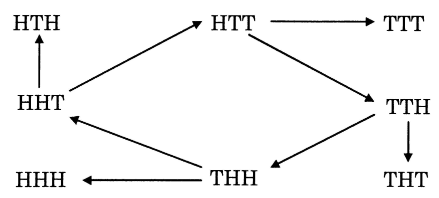

Figure 1: Relationship between the eight patterns.

relationship is shown in Figure 1 as a looped relationship among the four inner patterns.

Mathematically speaking, the following is referred to as a transitive relationship:

$$\text{If } A \Rightarrow B \text{ and } B \Rightarrow C \text{ then } A \Rightarrow C.$$

Rephrasing this in terms of a game, then we would have “ If player A beats player B and player B beats player C, then player A beats player C ”. But is this a true proposition? Gardner poses that this transitive relationship does not hold in the Penney Ante game, that it forms non-transitive relations [2]. Such three-way relationships may be unfamiliar in the West, but they are well known in Japan, for example in the old game of rock-scissors-paper. In that game, it does not follow that because rock beats scissors and scissors beat paper that therefore rock beats paper. Instead, rock loses to paper, therefore failing to uphold the transitive relationship. Similarly, transitive relationships do not hold in Penney Ante—as in rock-scissors-paper, there is no “ best play ” and the second player can always win.

The following rules exist for Player B’s selection, based on the selection made by Player A. Assume that Player A chooses HHH. In that case, the second call is for H. Flip that selection to a T, and move it to the beginning of Player B’s sequence. Next, take the first two calls in Player A’s sequence, and take those as the second and third call in Player B’s. Doing so results in THH for Player B, a selection that will win against Player A. Player A’s third call is not considered.

$$\underline{\mathrm{HH}}\mathrm{H} \rightarrow \mathrm{T}\underline{\mathrm{HH}}$$

Readers should confirm that each of the selections for Player B in Table 1 follow this rule. This rule for the selection by Player B based on the selection by Player A works for runs of length 3, but only for runs of length 3—the same algorithm will not necessarily work for other lengths.

Let us confirm the probabilities listed in Table 1. We will first calculate the probability that THH will appear before HHH. Assume that Player A selects HHH. Should that sequence appear in the first three tosses, then Player A will win. In any other situation, Player B will win. The probability that the first three tosses will come up HHH is $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$; thus,

$$P(\text{THH before HHH}){=}1 - \frac{1}{8} = \frac{7}{8}.$$

Player B's odds are therefore $\frac{7}{8} \div \frac{1}{8} = 7$, or 7:1.

Let us next calculate the probability that THH will appear before HHT. The probability of flipping HHT, or HHHT, or HHHHT$\cdots$ is

$$\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = \frac{1/8}{1-1/2} = \frac{1}{4}.$$

This gives the probability of HHT winning. The probability of THH winning is therefore the complementary event, and thus

$$P(\text{THH before HHT}){=}1 - \frac{1}{4} = \frac{3}{4}.$$

Player B's odds are therefore $\frac{3}{4} \div \frac{1}{4} = 3$, or 3:1.

Let us next calculate the probability that HHT will appear before HTH.

$$x{=}P(\text{HHT before HTH})$$

We can ignore all initial flips of T. Taking the first flip as H, if the next flip is also H then the probability of HHT appearing is 1/2. If a T is flipped, however, the sequence is reset and must wait for the next H flip. That then is equivalent to x. Shown as an equation, we have

$$x = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot x,$$

and solving for x gives

$$x = \frac{2}{3}.$$

Player B's odds are therefore $\frac{2}{3} \div \frac{1}{3} = 2$, or 2:1. Continuing in this manner, the individual odds for each pair can be calculated, but Conway's Algorithm provides a general solution. The following section describes that algorithm.

3. Conway's Algorithm

John Conway proposed an algorithm that can be used to easily calculate the odds of Player B beating Player A. Conway introduced leading numbers as an index indicating the degree of pattern overlap. Leading numbers are also an index of the level of repetition of a given pattern within a preceding pattern, and are called Conway Numbers after their inventor.

Let us now examine Conway's algorithm, taking as an example the case where Player A selected HHH, and Player B selected THH.

$$A - HHH$$

$$B - THH$$

Let us first find the Conway Number for Player A with regards to Player A's own selection. First, we place A–HHH over A–HHH. Next, if the HHH of the upper sequence is the same as the lower sequence, then we place a 1 over the first element and a 0 otherwise.

$$\begin{array}{l} \quad\;\; 1 \\ A-HHH \\ A-HHH \end{array}$$

Next, remove the leading H from the upper HHH sequence, and shift it to the left, aligning the leading elements. Then, compare the HH to the first two elements in the lower sequence, and if they are the same place a 1 above the leading element of the HH, or a 0 otherwise.

$$\begin{array}{l} 1 \\ HH \\ HHH \end{array} \qquad\qquad \begin{array}{l} 1 \\ H \\ HHH \end{array}$$

Repeat this procedure through to the last element of the upper sequence, and compile the results as follows.

$$\begin{array}{l} \quad\;\; 1\ 1\ 1 \\ A-HHH \\ A-HHH \end{array}$$

The resulting binary number 111 is an index of the overlap of A with regards to A, and the notation for the Conway number is AA=111. There are a total of $2 \times 2 = 4$ permutations of Conway numbers for a given pair of upper and lower sequences. The following are the Conway numbers for pairs *AA, AB, BB,* and *BA.*

$$\begin{array}{l} \quad\;\; 1\ 1\ 1\ =\ 7 \\ A-HHH \\ A-HHH \end{array} \qquad \begin{array}{l} \quad\;\; 0\ 0\ 0\ =\ 0 \\ A-HHH \\ B-THH \end{array}$$

$$\begin{array}{l} \quad\;\; 1\ 0\ 0\ =\ 4 \\ B-THH \\ B-THH \end{array} \qquad \begin{array}{l} \quad\;\; 0\ 1\ 1\ =\ 3 \\ B-THH \\ A-HHH \end{array}$$

Conway numbers are derived as binary numbers, but converting to decimal gives

$$AA = 7,\ AB = 0,\ BB = 4,\ BA = 3.$$

Using the four Conway numbers, the odds of Player B winning are given by the equation

$$(AA - AB)/(BB - BA).$$

Replacing the above four values in the equation, we have

$$\frac{AA - AB}{BB - BA} = \frac{7 - 0}{4 - 3} = 7.$$

Player B's odds are given as 7, matching with the first entry in Table 1.

Taking Player A's probability of winning as p, and Player B's probability of winning as q, we obtain

$$p = \frac{1}{1+7} = \frac{1}{8},\ q = \frac{7}{1+7} = \frac{7}{8}.$$

Both Player A and Player B have eight possible selections. Because the players make their selections independently, there are $8 \times 8 = 64$ possible matches, and the odds for each are automatically generated by the Conway algorithm as shown in Table 2. Taking the best odds for Player B with respect to Player A's selection gives the results listed in Table 1. Conway's algorithm is very powerful, in that it can give probabilities not only for sequences of length 3, but for those of any length, or even for sequences of dissimilar length.

		A							
		HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
	HHH		1/2	2/5	2/5	1/8	5/12	3/10	1/2
	HHT	1/2		2/3	2/3	1/4	5/8	1/2	7/10
	HTH	3/5	1/3		1/2	1/2	1/2	3/8	7/12
B	HTT	3/5	1/3	1/2		1/2	1/2	3/4	7/8
	THH	7/8	3/4	1/2	1/2		1/2	1/3	3/5
	THT	7/12	3/8	1/2	1/2	1/2		1/3	3/5
	TTH	7/10	1/2	5/8	1/4	2/3	2/3		1/2
	TTT	1/2	3/10	5/12	1/8	2/5	2/5	1/2	

Table 2. Probabilities of winning for players A and B (for sequences of length 3).

4. Proof according to average wait time

The algorithm for determining odds using Conway Numbers is indeed elegant and powerful. Let us consider precisely why one can calculate win probabilities from Conway Numbers. As elegant an algorithm as it is, Conway himself did not propose a proof in the paper in which he demonstrated it. Searching through the literature, I came across an easily understood proof that uses Stanley Collings's concept of average waiting time [3]. Collings's proof is developed from three theorems.

Theorem X
The expected waiting time to get a given sequence of integers from the set $\{1, 2, ..., k\}$ *is* $k.AA$*, where* AA *stands for the corresponding Conway Number considering entirely of 0's and 1's and interpreted in the scale of* k*.*

The faces of a coin $\{H, T\}$ can be mapped to $\{1, 2\}$. A–HHH, B–THH is therefore mapped to A–111, B–211, and their Conway Numbers are $AA = 111$, $BB = 100$. Converting the binary numbers to decimal we have $AA = 7$, $BB = 4$, and doubling them we have average wait times of 14 and 8.

Theorem Y
Given a sequence B to start with, the expected further number of rolls required to complete or produce the A sequence is $k.AA$–$k.BA$*.*

For example, given the sequence B–211, because $AA = 7$, $BB = 3$, the average wait time for A–111 to appear is $k.AA - k.BA = 2 \times 7 - 2 \times 3 = 8$.

Theorem Z
The odds that the B sequence precedes the A sequence are given by (AA-AB):(BB-BA).

When the average wait time of A is larger than that of B, then the difference is expressed as

$$k.(AA - BB).$$

On the other hand, from Theorem Y we have that the average wait time for A to appear when B precedes A is

$$k.AA - k.BA.$$

Similarly, the average wait time for B to appear when A precedes B is

$$k.BB - k.AB.$$

Taking the probability of Player A winning as p and the probability of Player B winning as q, q will occur with probability $k.AA - k.BA$, and p will occur with probability $k.BB - k.AB$, and so the three terms above are related by

$$k(AA - BB) = q(k.AA - k.BA) - p(k.BB - k.AB).$$

Dividing both sides of the equation by k, we obtain

$$AA - BB = q.AA - q.BA - p.BB + p.AB$$

and

$$(1-q)AA - (1-p)BB = p.AB - q.BA.$$

Because $p + q = 1$, we have

$$p.AA - q.BB = p.AB - q.BA,$$
$$p(AA - AB) = q(BB - BA),$$

and

$$\frac{q}{p} = \frac{AA - AB}{BB - BA}.$$

The last equation gives the odds of Player B beating Player A, and proves Conway's algorithm.

5. Application to a card game

In the preceding sections, we examined Penney's Ante using heads and tails in a coin flipping game; however, Steve Humble and I have been discussing how this might be applied to a more familiar context, such as a card game. While coin tosses give equal probabilities of heads or tails, we cannot guarantee that the probabilities are exactly 1/2. The situation is the same with the red/black results of a roulette table. Coin flips are also problematic in that the action of flipping a coin is somewhat difficult, and one must maintain a record of the results. We therefore turned our consideration to packs of cards.

There are 52 cards in a deck, and the 26 black (spades and clubs) and 26 red (hearts and diamonds) cards can be used to substitute for the heads or tails results of a coin flip. Let B represent black cards, and R red cards, so that HHH versus THH when flipping coins could be represented as BBB versus RBB when using cards.

The advantage of using cards lies in their ease of manipulation, and the fact that one does not need to keep track of results. They can also be well randomized. If all 26 red and black cards are used, then there is a 1/2 chance of choosing either color from the 52 total cards. The 52 cards also allow an appropriate number of trials. It is difficult to determine the end of a game that uses coin flips, but one can simply declare an end when all 52 cards are used. The only problem to consider is if they will form a game. The number of trials that can be performed with 52 cards is given by Collings's theory [3], as shown below, which states that the average wait time *E(N)* before sequence A or B will appear is

$$E(N) = k(p.AA + q.BA)$$

or

$$E(N) = k(p.AB + q.BB).$$

The average wait time before A or B appears can be given as a group $A \cup B$.

As shown in Table 3, the values of *E(N)* differ according to matches, giving 7 for BBB versus RBB, and 16/3 for BRR versus BBR. Here, the average wait time *E(N)* refers to the number of tricks played per match. Dividing this number by the number of cards (52) therefore gives the number of trials. The theoretical values give a distribution of 7.4–9.8 trials, which matches well with the statistical values we found when we tried this with actual cards (7–10).

A	B	AA	BB	AB	BA	p	q	$E(N)$	Trials
BBB	RBB	7	4	0	3	1/8	7/8	7	7.4
BBR	RBB	4	4	1	3	1/4	3/4	13/2	8
BRB	BBR	5	4	1	2	1/3	2/3	6	8.7
BRR	BBR	4	4	0	2	1/3	2/3	16/3	9.8
RBB	RRB	4	4	0	2	1/3	2/3	16/3	9.8
RBR	RRB	5	4	1	2	1/3	2/3	6	8.7
RRB	BRR	4	4	1	3	1/4	3/4	13/2	8
RRR	BRR	7	4	0	3	1/8	7/8	7	7.4

Table 3. Average trials using a pack of cards.
(*E(N)*: Average wait times)

From Table 3 we see that the probabilities for BBB and RBB are 1/8 and 7/8, giving RBB overwhelming odds of 7:1 in a match versus BBB. In the match RBR versus RRB, however, the odds for RRB are only 2:1. Such odds imply no guarantee of winning in a single match. We would therefore like to perform n trials to increase the odds of RRB winning. For example, in a match of three tricks, Player B will win by taking two tricks, and in a match of seven tricks Player B will win by taking four. Showing this as an equation gives us the following. Taking $P(\text{RRB}, n)$ over n trials, the probability of an RRB win is

$$P(\text{RRB},7)= {}_7\mathrm{C}_7\left(\frac{2}{3}\right)^7 + {}_7\mathrm{C}_6\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right) + {}_7\mathrm{C}_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2 + {}_7\mathrm{C}_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^3 = 0.827,$$

a clear advantage for Player B over the 0.667 probability for a single trial. Playing the game using cards over 7–9 trials, we can expect a very high probability that Player B will win.

We have shown in this article that Penney's game can be applied to the use of cards, and that cards are better suited to its application than coin tosses. We would like to present this new card-based version as the **Humble-Nishiyama Randomness Game**, and hope that our readers will try it with a deck of cards at home.

References

[1] Walter Penney, 95. Penney-Ante, *Journal of Recreational Mathematics*, **2**, (1969), 241.

[2] Martin Gardner, Mathematical Games, *Scientific American*, October 1974, **231**, No.4, 120-125.

[3] Stanley Collings, Coin Sequence Probabilities and Paradoxes, *Bulletin of the Institute of Mathematics and its Applications*, **18**, November/December 1982, 227-232.