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Strong Convexity of NTU Games

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Abstract

In this paper, we provide an alternative definition of NTU convexity, *strongly ordinal convexity*. We show that if a game is strongly ordinal convex, then any marginal worth vector is in the core, and any marginal contribution is increasing. Some economic examples satisfy this convexity.

KeywordsCooperative game; Convex game; NTU game; Core; Supermodularity

JEL classification codes: C62; D52; D53

1 Introduction

The convexity of NTU game was first defined as an extension of the TU convex game by Vilkov (1977), called *weakly ordinal convexity* in this paper. While weakly ordinal convex games arise in various economic applications[1], they do not inherit interesting properties from the TU convex game, any of which also characterizes the TU convexity (see Sharkey, 1981; Hendrickx et al. 2000, 2002). Moreover, these properties do not imply the weakly ordinal convexity. In this paper, I propose a strong concept of NTU convexity, *strongly ordinal convexity*, to inherit the various properties of the TU convex games. Roughly speaking, a game is strongly ordinal convexity iff for all coalition T, the proper contribution of coalition $S \subset N \setminus T$ in $S \cup T$ is super additive with respect to S. We show that a strongly ordinal convex game is weakly ordinal convex, any marginal contribution of this game is increasing, and its core comprises all marginal worth

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[1]See Peleg (1984).

vectors. Further, we show that some economic examples are strongly ordinal convex.

2 Results

2.1 Preliminary

Let $\Re$ be the set of real numbers. Let $N = \{1, 2, \ldots, n\}$ be a finite set of players. A nonempty subset of N is called a coalition. A payoff vector is an element in $\Re^N$. For all $S \in 2^N$ and $x \in \Re^N$, x^S denotes the projection of x to $\Re^S$, and $x(S)$ denotes $\sum_{i \in S} x_i$. For all $x, y \in \Re^N$, we write $x^S \geq y^S$ (resp. $x^S \gg y^S$) iff $x^i \geq y^i$ (resp. $x^i > y^i$) for all $i \in S$. A *coalitional form game* specifies a set of payoffs that can be obtained by coalition S by itself. A *TU coalitional form game*, v, is a function from 2^N to $\Re$ with $v(\emptyset) = 0$. A payoff $x \in \Re^N$ can be obtained by S by itself iff $x(S) \leq v(S)$. A *core* of v is a set of payoff vectors $x \in \Re^N$ such that $x(S) \geq v(S)$ for all $S \in 2^N$ and $x(N) \leq v(N)$. A game v is *convex* if $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$ for all $S, T \in 2^N$. This definition is equivalent to the property, called *the increasing marginal contribution*, that $v(Q \cup R) - v(Q) \leq v(T \cup R) - v(T)$ for all $Q \subseteq T \subseteq N \setminus R$. Let σ be a permutation of N. The *marginal worth vector* for σ is a payoff vector, denoted by m_σ, such that $m_\sigma^{\sigma(1)} = v(\sigma(1))$; for all $k > 1$, $m_\sigma^{\sigma(k)} = v(\sigma(\{1, 2, \ldots, k\})) - v(\sigma(\{1, 2, \ldots, k-1\}))$. Shapley (1971) and Ichiishi (1981) showed that a TU game is convex iff any marginal worth vector is in the core.

An NTU game is a correspondence from 2^N to $\Re^N$ such that for all $x, y \in \Re^N$, if $x^S \geq y^S$ and $x \in V(S)$, then $y \in V(S)$, and that $V(\emptyset) = \emptyset$. Any situation described by a TU game v is also expressible by an NTU game such that $V_v(S) := \{x \in \Re^N \mid v(S) \geq x(S)\}$. For all $S \in 2^N$, let $D(S)$ denote the set of payoff vectors x such that there exists no $y \in V(S)$ such that $y^S \gg x^S$. Let $\hat{V}(S)$, and $\hat{D}(S)$ denote the projections of $V(S)$ and $D(S)$ to $\Re^S$ respectively. A core of V is defined by $V(N) \bigcap (\cap_{S \subseteq N} D(S))$.

An NTU game is *weakly ordinal convex* if for all $S, T \in 2^N \setminus \{\emptyset\}$, $V(S) \cap$

$V(T) \subseteq V(S \cup T) \cup V(S \cap T)$. This notion is an extension of the TU convexity to NTU game in the sense that V_v is weakly ordinal convex iff v is convex. The weakly ordinal convexity playes important roles in the effectivity function analayses (see chapter 6 of Peleg 1984). Further, the core of weakly ordinal convex game is nonempty and coincides with the unique vNM stable set (see Greenberg,1985; Peleg, 1986).

2.2 Strong Convexity

For all $R \in 2^N \setminus \{\emptyset\}$ and all $Q \subseteq N \setminus R$, x^R is a *proper contribution* of R to x^Q iff $x \in V(R \cup Q) \cap D(Q)$. That is, x^R is a payoff that can be allocated to R if R collaborates with Q and gives x^Q to Q, which cannot be improved upon by Q by itself. Now, we define our main concept:

Definition 1 *An NTU game is strongly ordinal convex if*

$$V(S) \cap D(S \cap T) \cap V(T) \subseteq V(T \cup S) \text{ for all } S, T \in 2^N.$$

By taking $R_0 := S \cap T$, $R_1 := S \setminus T$, and $R_2 := T \setminus S$, one can easily see that a game is strongly ordinal convex iff proper contributions are super-additive with respect to coalitions: for all $R_0, R_1, R_2 \subseteq N$ such that $R_i \cap R_j = \emptyset$ for all $i \neq j$, if both of x^{R_1} and x^{R_2} are proper contributions of R_1 and R_2 to x^{R_0} respectively, then (x^{R_1}, x^{R_2}) is a proper contribution of $(R_1 \cup R_2)$ to x^{R_0}. This condition was first proposed by Milgrom and Shannon (1996), who attempted to define a general notion of NTU convexity[2].

First, we show that this definition is an alternative extension of the TU convexity.

Theorem 2 *For any TU game v, an NTU game V_v defined by $V_v(S) := \{x \in \Re^N \mid v(S) \geq x(S)\}$ for all $S \in 2^N \setminus \{\emptyset\}$ is strongly ordinal convex iff v is convex.*

[2]Their generalization is not successful as Theorem 8 of their paper is mathematically incorrect.

Proof. For all $x \in \Re^N$, $x \in V_v(S) \cap D_v(S \cap T) \cap V_v(T)$ iff $-x(S \cap T) \leq -v(S \cap T)$, $x(S) \leq v(S)$, and $x(T) \leq v(T)$. Then, for all $x \in V_v(S) \cap D_v(S \cap T) \cap V_v(T)$,

$$v(S) + v(T) - v(S \cap T) \geq x(S \cup T). \tag{1}$$

If v is convex, then the left hand side of (1) is smaller than or equal to $v(S \cup T)$, which implies that $x \in V_v(S \cup T)$. Conversely, suppose that V_v is strongly ordinal convex. We can choose $x \in V_v(S) \cap D_v(S \cap T) \cap V_v(T)$ to satisfy the equality of (1).The right hand side of (1) is then smaller than or equal to $v(S \cup T)$. It follows that v is convex.□

Our convexity implies the convexity mentioned in Vilkov (1977).

Theorem 3 *A strongly ordinal NTU game is weakly ordinal convex.*

Proof. Suppose that V is strongly ordinal convex. Let $x \in V(S) \cap V(T)$. If $x \in D(S \cap T)$, then we have $x \in V(S \cup T)$. On the other hand, if $x \notin D(S \cap T)$, then there exists $y \in V(S \cap T)$ such that $y^{S \cap T} \gg x^{S \cap T}$ and, therefore, $x^{S \cap T}$ is also in $\hat{V}(S \cap T)$.□

The *marginal worth vector* for σ, m_σ, is defined by $m_\sigma^{\sigma(k)} := \max\{y^{\sigma(k)} \mid (m_\sigma^{\sigma(\{1,2,\ldots,k-1\})}, y^{\sigma(k)}) \in \hat{V}(\sigma(\{1, 2, \ldots, k\}))\}$. Hendrickx et al. (2000, 2002) illustrated that the core of an ordinal convex game does not necessarily comprise all marginal worth vectors. On the other hand, we have the following:

Theorem 4 *The core of a strongly ordinal NTU game comprises all marginal worth vectors.*

Proof. Consider the induction on $|N|$. Let σ be a permutation and W be a restriction of V to 2^T, where $T := N \setminus \{\sigma(n)\}$. Suppose that there exist $S \neq \emptyset$ and $y \in \Re^N$ such that $y \in V(S)$ and $y^S \gg m_\sigma^S$. By the induction hypothesis, m_σ^T is in the core of W. Thus, $\sigma(n) \in S$ and $m_\sigma \in D(S \cap T)$. Then, $(m_\sigma^T, y^{\sigma(n)}) \in V(S) \cap D(S \cap T) \cap V(T)$. From strongly ordinal convexity, $(m_\sigma^T, y^{\sigma(n)}) \in V(N)$; this contradicts the definition of $m_\sigma^{\sigma(n)}$.□

Economic Examples Economic models in Masuzawa (2004) satisfy the condition that $V(S) \cap V(T) \subseteq V(S \cup T)$ for all $S, T \in 2^N$. Obviously, they are strongly ordinal convex. Note that this condition is not adequate for an alternative concept of NTU convexity since V_v does not satisfy this condition for all TU games v.

Weakly ordinal convex games arise in voting theory[3]. On the other hand, we have the following example.

Example 5 Consider a three-person majority voting game with two alternatives, a and b. Suppose that a and b give the players $(x^1, x^2, x^3) = (1, 0, 0)$ and $(y^1, y^2, y^3) = (0, 1, 0)$ respectively. Then,
$V(\{i\}) = \{x \mid x^i \leq 0\}$ for $i = 1, 2, 3$;
$V(\{i, 3\}) = \{x \mid (x^i, x^3) \leq (1, 0)\}$ for $i = 1, 2$;
$V(\{1, 2\}) = \{x \mid \max\{x^1, x^2\} \leq 1 \text{ and } \min\{x^1, x^2\} \leq 0\}$;
$V(\{1, 2, 3\}) = \{x \mid x^3 \leq 0 \text{ and } x \in V(\{1, 2\})\}$.
This is an weakly ordinal convex game that is not strongly ordinal convex.

2.3 Increasing marginal contribution

For all $R \subseteq N$ and $Q \subseteq N \setminus R$, x^R is a *marginal contribution* to x^Q if it is a proper contribution of R to x^Q and $x^Q \in \hat{V}(Q)$. I show that any marginal contribution of a strongly ordinal convex game is increasing in Q. By taking $Q := S \cap T$ and $R := S \setminus Q$, we can see that an NTU game is strongly ordinal convex iff $V(R \cup Q) \cap D(Q) \cap V(T) \subseteq V(T \cup R)$ for all $Q \subseteq T \subseteq N \setminus R$. It follows that for all $Q \subseteq T \subseteq N \setminus R$,

$$V(R \cup Q) \cap \big(D(Q) \cap V(Q)\big) \cap \big(D(T) \cap V(T)\big) \subseteq V(T \cup R). \qquad (2)$$

Condition (2) says that if x^R is a marginal contribution to x^Q, then it is also a marginal contribution to all $y^T \in \hat{D}(T) \cap \hat{V}(T)$ such that $y^Q = x^Q$. Note that for any TU game v, V_v satisfies (2) iff it satisfies the increasing marginal contribution property: $v(Q \cup R) - v(Q) \leq v(T \cup R) - v(T)$.

[3]See chapter 6 of Peleg (1984).

Note that in some weakly ordinal convexity games, a marginal contributions are not increasing in our sense.

Example 6 Consider a three-person game defined by
$V(\{i\}) = \{x \mid x^i \leq 0\}$ for $i = 1, 2, 3$;
$V(\{1, 2\}) = \{x \mid x^1 + x^2 \leq 1\}$;
$V(\{i, 3\}) = \{x \mid (x^i, x^3) \leq (0, 1)\}$ for $i = 1, 2$;
$V(\{1, 2, 3\}) = \{x \mid x^1 + x^2 + x^3 \leq 1\}$.
This game is weakly ordinal convex. However, the marginal contribution of $\{3\}$ to $x^1 = 0$ is 1 while that of $\{3\}$ to $y^{\{1,2\}} \in \hat{V}(\{1, 2\}) \cap \hat{D}(\{1, 2\})$ is 0.

A payoff vector a^R is *acceptable for* $Q \subseteq N \setminus R$ iff it is a proper contribution to some a^Q. Milgrom and Shannon (1996) introduced the following property, called *the scale merit of acceptability* in this paper: for all $Q \subseteq T \subseteq N \setminus R$ and all $a^R \in \Re^R$, if a^R is acceptable for Q, then a^R is also acceptable for T.

Theorem 7 *Let V be an NTU game. Assume that for all $S \in 2^N \setminus \{\emptyset\}$, $V(S)$ is closed and nonempty, and that for all $b \in V(S)$, $\{x^S \in \hat{V}(S) \mid x^S \geq b^S\}$ is bounded in $\Re^S$. If V is strongly ordinal convex, then V has the scale merit of acceptability.*

Proof. It suffices to consider the case where $Q \subsetneq T = N \setminus R$. Suppose that a^R is acceptable for Q. Then, there exists $a^Q \in \hat{D}(Q)$ such that $(a^R, a^Q) \in \hat{V}(Q \cup R)$. Since $V(Q)$ is nonempty, there exists $x^Q \in \hat{V}(Q)$ such that $x^Q \leq a^Q$. Since $V(Q)$ is closed, there exists $b^Q \in \hat{D}(Q) \cap \hat{V}(Q)$ such that $(a^R, b^Q) \in \hat{V}(R \cup Q)$. From strongly ordinal convexity and $V(T \setminus Q) \neq \emptyset$, there exists $y^{T \setminus Q} \in \Re^{T \setminus Q}$ such that $(y^{T \setminus Q}, b^Q) \in \hat{V}(T)$. Since $\{x^T \in \hat{V}(T) \mid x^T \geq y^T\}$ is bounded and closed, we can choose $c^{T \setminus Q}$ such that $(b^Q, c^{T \setminus Q}) \in \hat{V}(T) \cap \hat{D}(T)$. Then, $(a^R, b^Q, c^{T \setminus Q}) \in V(R \cup Q) \cap D(Q) \cap V(T)$. From strongly ordinal convexity, $(a^R, b^Q, c^{T \setminus Q}) \in V(T \cup R)$. Further, since $(a^R, b^Q, c^{T \setminus Q}) \in D(T)$, a^R is also acceptable for T.□
The converse of Theorem 7 is, however, not true.

Example 8 Consider a three-person game such that

$V(\{i\}) = \{x \mid x^i \leq 0\}$ for all $i = 1, 2, 3$;

$V(S) = \{x \mid \min_{i \in S}\{x^i\} \leq 1 \text{ and } \max_{i \in S}\{x^i\} \leq 2\}$ if $|S| = 2$;

$V(S) = \{x \mid x(S) \leq 3 \text{ and } \min_{i \in S}\{x^i\} \leq 0\}$ if $|S| = 3$.

This game satisfies the scale merit of acceptability. The core of the game is, however, empty.

Hendrickx et al.(2000, 2002) proposed *the coalition-merge property*: for all $Q \subseteq T \subseteq N \setminus R$, all $a \in D(Q) \cap V(Q \cup R) \cap \left(\cap_{i \in Q} D(\{i\})\right)$, and all $b \in V(T)$, $(a^R, b^{N \setminus R}) \in V(T \cup R)$. This requirement is strong in that a^R is compatible with all $b^T \in \hat{V}(T)$, which is chosen independently of a^Q. On the other hand, note that, in our definition, b^T is assumed to be an extension of a^Q.

Example 9 Consider a four-person game defined by

$V(\{1, 2, 3, 4\}) = \{x \mid x(\{1, 2, 3, 4\}) \leq 2\}$;

$V(\{1, 2, 3\}) = \{x \mid (x^1, x^2, x^3) \leq (1, 0, 1)\}$;

$V(\{1, 2, 4\}) = \{x \mid (x^1, x^2, x^4) \leq (0, 1, 1)\}$;

$V(S) = \{x \mid x^i \leq 0 \text{ for all } i \in S\}$ otherwise.

This game does not satisfy the coalition-merge property. Let $Q = \{1, 2\}$, $R = \{3\}$, $T = \{1, 2, 4\}$. Then, $(a^1, a^2, a^3, x) = (1, 0, 1, x) \in D(Q) \cap V(Q \cup R) \cap \left(\cap_{i \in Q} D(\{i\})\right)$ and $(b^1, b^2, b^3, b^4) = (0, 1, 0, 1) \in V(T)$. However, $(a^R, b^T) = (b^1, b^2, a^3, b^4) = (0, 1, 1, 1) \notin V(T \cup R)$. On the other hand, it is strongly ordinal convex.

The following example from Hendrickx et al. (2000, example 4.6) also illustrates that the coalition-merge property does not necessarily mean the properties discussed in this paper:

Example 10 Consider a four-person game defined by

$V(\{1, 2, 3, 4\}) = \{x \mid x(\{1, 2, 3, 4\}) \leq 7\}$;

$V(S) = \{x \mid x(S) \leq 4\}$ if $|S| = 3$;

$V(S) = \{x \mid \max_{i \in S} x^i \leq 1\}$ if $|S| = 2$;

$V(S) = \{x \mid x^i \leq 0 \text{ for all } i \in S\}$ otherwise.

Hendrickx et al. (2000) showed that this game has the coalition-merge property but is not weakly ordinal convex. It does not satisfy the scale merit of acceptability. To see this, consider $x^1 = 4$, which is a proper contribution of player 1 to $(x^2, x^3) = (1, -1)$. It is not, however, a proper contribution of player 1 to any (y^2, y^3, y^4), which must be smaller than 3. It follows that the coalition merge property does not imply strong convexity. Further, $(x^1, x^2, x^3, x^4) = (4, 1, -1, 4) \in V(\{1,2,3\}) \cap V(\{2,3\}) \cap D(\{2,3\}) \cap V(\{2,3,4\}) \cap D(\{2,3,4\})$ while $(x^1, x^2, x^3, x^4) \notin V(\{1,2,3,4\})$. Thus, a marginal contribution is not increasing in our sense.

3 Concluding Remarks

We defined strongly ordinal convexity as an extension of the TU convexity to NTU game, which is a subclass of the weakly ordinal convex games of Vilkov (1977). Further, any marginal contribution of this game is increasing and the core of this game comprises all marginal worth vectors.

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