Noncooperative Foundation of n-Person Asymmetric Nash Bargaining Solution

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Abstract

This paper presents a noncooperative foundation of the asymmetric Nash bargaining solution for an n-person bargaining problem. We show that an SSPE (stationary subgame-perfect equilibrium) payoff vector of the noncooperative bargaining model is equal to the asymmetric Nash bargaining solution of the bargaining problem as the risk of the breakdown of negotiations is very small. If the feasible set of payoffs is convex in the bargaining problem, any Pareto-efficient payoff allocation is implemented by an appropriately given bargaining procedure. In this, there is a one-to-one correspondence between the weight of players for the asymmetric Nash bargaining solution and the probability distribution for selecting a proposer in the noncooperative bargaining game.

JEL Classification Numbers: C72, C78.

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1 Introduction

We present a model of noncooperative bargaining to support the n-person Nash bargaining solution. More precisely, it is shown that as the probability of the breakdown of negotiations goes to zero (that is, as the cost of delay becomes small), payoff vectors in stationary subgame-perfect equilibria of our bargaining game converge to the n-person asymmetric Nash bargaining solution. The program of establishing noncooperative foundations for the Nash bargaining solution (characterized by four axioms in Nash, 1950) was initiated by Nash (1953). He provided a model of simultaneous offers, called Nash's "demand game", and showed that the Nash bargaining solution of a 2-person bargaining problem is supported by a Nash equilibrium that is robust to perturbations in the structure of the game. Subsequently, Rubinstein (1982) presented an alternating-offer bargaining game where the payoff vectors in every subgame-perfect equilibrium converge to the Nash bargaining solution as the players' discount factor goes to one. Both of these papers study the bilateral bargaining problem and the symmetric Nash bargaining solution.

The extensions to the n-person multilateral bargaining problem has been pursued by Hart and Mas-Colell (1996) and Krishna and Serrano (1996). They presented a model of noncooperative bargaining game to implement the *symmetric* Nash bargaining solution of an n-person bargaining problem. Our bargaining procedure is a variation of the random proposer model (Hart and Mas-Colell, 1996, Okada, 1996). In the model, one player is selected as a proposer according to some probability distribution among n players and proposes a feasible payoff vector. The proposal is agreed to by unanimous consent among the players and the game then ends with these payoffs. If it is rejected by some players, with an almost all probability the same bargaining game is repeated again, but with a small probability negotiations break down. In Hart and Mas-Colell's procedure, a proposer is randomly selected with equal probability among all players. In the case of the pure bargaining problem (that is, problems where the only possible final outcomes are either the full cooperation of all players or the complete breakdown of cooperation), our bargaining procedure provides a generalized model of Hart and Mas-Colell. Krishna and Serrano (1996) also provided a multilateral bargaining game different from the one that requires unanimous agreement. In their model, the proposal protocol is predetermined, and players who accept the proposal receive their payoffs immediately and exit the game. Negotiations then continue among the players who rejected the proposal.

A bargaining problem can be described by a pair (V, r), where V is the feasible set of utilities and r is a disagreement point. Note that our bargaining procedure generates the *asymmetric* Nash bargaining solution of the bargaining problem (V, r): that is, it generates the solution of the following maximization problem: $\max_{v \in V} \prod_{i \in N} (v_i - r_i)^{\theta_i}$, where $\sum_{i \in N} \theta_i = 1$ and $\theta_i > 0$. The symmetric Nash bargaining solution is regarded as a special case of $\theta_i = 1/n$ for all $i \in N$. Owing to this generalization, we are also able to show that if the feasible set V is convex, any Pareto-efficient payoff allocation is realized through our bargaining procedure by choosing an appropriate probability distribution in selecting a proposer. This is the second result of our paper.

Recently, Okada (2005) presented a noncooperative foundation of the asymmetric Nash bargaining solution for a cooperative game in strategic form. His bargaining procedure is almost the same as that in the current paper. Since a strategic form game was considered, players would negotiate their actions in the strategic form game. Moreover, players are permitted to form coalitions. In current model, players negotiate the allocation of payoffs and are prohibited from strategically forming coalitions. However, because of the restriction to pure bargaining situations, we obtain more clearcut results than those derived in Okada[1].

This paper is organized as follows. Section 2 defines the asymmetric Nash bargaining solution of the n-person bargaining problem. Section 3 provides our noncooperative bargaining game model and defines the equilibrium concept. Section 4 states the main results. Section 5 gives the proof of the Theorem 1.

2 Nash Bargaining Solution

Let us define the bargaining problem. The set of n players is $N = \{1, \ldots, n\}$ $(n \geq 2)$, and the feasible set of utilities is denoted by V, which is a subset of the n-dimension Euclidean space $\mathbb{R}^n$. A disagreement point is $r = (r_i)_{i \in N} \in V$, where it is assumed that $r_i \geq 0$. The bargaining problem consists of the feasible set V and a disagreement point r. We make the following assumptions about V.

Assumption 1. The set V is closed, convex and comprehensive, and V contains a point $y = (y_i)_{i \in N}$ such that $y_i > r_i$ for $i = 1, 2, \ldots, n$. Moreover, $V \cap \mathbb{R}^n_+$ is assumed to be bounded. The boundary $\partial V \cap \mathbb{R}^n_+$ is smooth and nonlevel.

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[1]Miyakawa (2003) has already shown that the (symmetric) Nash bargaining solution (equivalently, the maximum solution of the Nash social welfare function) is supported by the limit of equilibria of our noncooperative bargaining procedure in the pure bargaining case. The connections between the Nash bargaining problem and the Nash social welfare function have been investigated in Kaneko (1980).

Note that $\partial V \cap \mathbb{R}^n_+$ is smooth if and only if at each $y \in \partial V \cap \mathbb{R}^n_+$, there exists a single outward normal direction. In addition, $\partial V \cap \mathbb{R}^n_+$ is nonlevel if and only if the outward normal vector at any point of $\partial V \cap \mathbb{R}^n_+$ is positive in all coordinates. If $\partial V \cap \mathbb{R}^n_+$ is smooth and nonlevel, there exists a continuous, concave and differentiable function on $\mathbb{R}^n$ such that $H(y) = 0$ for all $y \in \partial V \cap \mathbb{R}^n_+$ and that $H(y) \geq 0$ for all $y \in V \cap \mathbb{R}^n_+$.

The set V can be interpreted as the feasible set of von Neumann Morgenstern utilities in that n individuals are derived from preferences over lotteries that satisfy the expected utility assumption. If we assume that randomization on different bargaining outcome is possible and that individuals can freely dispose of utility, both the convexity and the comprehensiveness assumptions are naturally satisfied. The smoothness and nonlevelness assumptions are technical. On the other hand, V also can be interpreted as the set of non-expected utility levels that they can reach in a particular bargaining situation. For example, consider situations where n individuals bargain over the division of a pie of size E. The set of possible agreement is $X = \{(x_i)_{i \in N} \in \mathbb{R}^n_+ \mid \sum_{i \in N} x_i = E\}$, where x_i is the share of the pie to player i. For each x_i, $u_i(x_i)$ is player i's utility from obtaining x_i, where u_i is a strictly increasing function. Then, the feasible set of utilities is defined as

$$V = \{(y_i)_{i \in N} \in \mathbb{R}^n_+ \mid \exists (x_i)_{i \in N} \in X, \forall i \in N, y_i \leq u_i(x_i)\}.$$

If the utility function u_i is continuous, concave and differentiable, the corresponding V satisfies the above assumptions; i.e., V is closed, convex, comprehensive, smooth and nonlevel.

For each bargaining problem (V, r), we can define the Nash bargaining solution as follows.

Definition 1. Let (V, r) be an n-person bargaining problem, and $\theta = (\theta_i)_{i \in N}$ satisfying $\sum_{i \in N} \theta_i = 1$ and $\theta_i > 0$. A payoff vector y^* is called the *asymmetric*

Nash bargaining solution of (V, r) with the weight vector θ if y^* is a solution of the maximization problem:

$$\max_{y \in V} \prod_{i \in N} (y_i - r_i)^{\theta_i}.$$

If $\theta_i = 1/n$ for all $i = 1, 2, \ldots, n$, the solution y^* represents the (symmetric) Nash bargaining solution; thus, $\prod_{i \in N}(y_i^* - r_i) \geq \prod_{i \in N}(y_i - r_i)$ for all $(y_i)_{i \in N} \in V$.

Using the notion of function H, we can rewrite the maximization problem in Definition 1 as

$$\max_{y} \prod (y_i - r_i)^{\theta_i} \text{ subject to } \; H(y) \geq 0 \text{ and } y_i \geq r_i, \; \forall i \in N.$$

By assumptions of V, the solution y^* satisfies $H(y^*) = 0$ and $y_i^* > r_i$ for all $i \in N$. Then, we obtain the Kuhn-Tucker condition from the maximization problem:

$$\frac{\theta_j}{y_j - r_j} \prod_{i \in N} (y_i - r_i)^{\theta_i} - \lambda \frac{\partial H}{\partial y_j}(y^*) = 0, \; j = 1, \ldots, n, \tag{1}$$

$$H(y^*) = 0, \tag{2}$$

where λ is the Lagrange multiplier. From assumptions of V and the concavity of $H(y)$ (the convexity of V), it follows that y^* is the solution of the maximization problem if and only if y^* satisfies the Kuhn-Tucker condition (1), (2).

3 Noncooperative Bargaining

We describe a noncooperative bargaining procedure to yield the n-person asymmetric Nash bargaining solution as the (limit of) equilibria. Let $0 \leq$

$\rho < 1$ be a fixed parameter. Then the n-person noncooperative bargaining model runs as follows.

(i) At the beginning of each round $t = 1, 2, \ldots$, one player is selected as a proposer according to the probability distribution θ. In other words, every player i is randomly chosen as a proposer with probability θ_i. The selected player i proposes a feasible payoff vector in V.

(ii) All other players in N either accept or reject the proposal sequentially. The responses are made according to a predetermined order over N. If all members of $N \backslash \{i\}$ accept, then the game ends with these payoffs. If some members of $N \backslash \{i\}$ reject, then the game moves to the next round. With probability ρ, negotiations continue in the next round with N under the same rule as in round t, thus, the game returns to (i). With probability $1 - \rho$, negotiations break down and the game ends. Each player i gets a payoff of r_i.

Every player has perfect information about the history of the game play. We do not consider a time discount. In place of a time discount in our bargaining model, the probability ρ represents the cost of delay in agreement. If $\rho \to 1$, the cost of delay is very low.

Our bargaining model can be represented as an infinite-length extensive form game with perfect information and with chance moves. We denote with $G^\theta(\rho)$ the bargaining model with a parameter ρ and a probability distribution θ. A strategy for player i in $G^\theta(\rho)$ is a sequence $\sigma_i = \{\sigma_i^t\}_{t=1}^\infty$ of mappings, where σ_i^t is the tth round strategy. The tth round strategy σ_i^t prescribes a proposal $y_i^t \in V$ as a proposer and a response function assigning "accept" or "reject" to all possible proposals by other players. For a strategy combination $\sigma = (\sigma_1, \ldots, \sigma_n)$, the expected payoffs for the players in $G^\theta(\rho)$ are determined in the usual manner. We use G^θ to describe the bargaining model where

$\rho \to 1$.

We apply a stationary subgame-perfect equilibrium as a solution concept to the bargaining model $G^\theta(\rho)$.

Definition 2. A strategy combination $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ of $G^\theta(\rho)$ is said to be a *stationary subgame-perfect equilibrium* (**SSPE**) if it is a subgame-perfect equilibrium where every player's strategy is limited to be stationary, i.e., for every round $t = 1, 2, \ldots$, the tth round strategy of every player does not depend on any history until round $t-1$ and only depends on history within round t.

The concept of an SSPE has been employed in almost all of the literature concerning the noncooperative multilateral bargaining model: see Chatterjee et al. (1993), Hart and Mas-Colell (1996), Okada (1996) among others. It is well known that in the sequential multilateral bargaining games, many payoff allocations are supported by subgame-perfect equilibria. For example, Osborne and Rubinstein (1990) showed the multiplicity of subgame-perfect equilibria in the 3-person sequential bargaining model. Assuming the stationarity, we can avoid the multiplicity problem of subgame-perfect equilibria in the n-person noncooperative bargaining model.

4 Main Theorems

In this section, we state our main theorems. The following theorem states that when the delay of cost converges to zero; $\rho \to 1$, the payoff vector at the asymmetric Nash bargaining solution with weight θ is realized by the limit of SSPEs of G^θ. We can implement the generalized n-person Nash bargaining solution in a noncooperative manner.

Theorem 1. *Let (V, r) be an n-person bargaining problem. Then, for each ρ; $0 \leq \rho < 1$, there exists an SSPE of the game $G^\theta(\rho)$. Moreover, any SSPE payoff vector $v(\rho)$ converges to the asymmetric Nash bargaining solution of (V, r) with the weight vector θ as $\rho \to 1$.*

The formal proof of Theorem 1 follows in the next section. We now give a sketch of the proof. The existence of an SSPE can be proved straightforwardly by the fixed point theorem. We next present a "simple" SSPE strategy for every player. In the SSPE, a proposer offers the other players their continuation payoffs and obtains the maximum residual payoff herself, and a responder accepts a proposal if and only if she is herself offered at least a certain continuation payoff. Using this SSPE, we derive an equations system that is satisfied by the expected equilibrium payoff vector satisfies. By virtue of the fact that the dispersion among individual proposals would vanish as $\rho \to 1$, we can finally confirm that the limit of SSPE payoff vectors as $\rho \to 1$ satisfies the Kuhn-Tucker condition of the maximization problem to obtain the Nash bargaining solution.

Remark a (one to one) correspondence between the weight parameter θ in the Nash bargaining solution and the probability distribution θ in the noncooperative bargaining procedure. The asymmetric Nash bargaining solution with the weight vector $\theta = (\theta_1, \ldots, \theta_n)$ is generated as the limit of SSPE payoff vectors in the noncooperative bargaining game $G^\theta(\rho)$, where each player i is randomly recognized to make a proposal with probability θ_i, when ρ converges to one.

Let us next discuss the relationship with the Pareto-efficient allocation.

We can define the Pareto-efficient payoff allocation of (V, r) by

$$\begin{aligned}\text{PEP}(V) = \{v \in V \mid \ &\text{there is no } v' \in V \text{ such that} \\ &v'_i \geq v_i \text{ for all } i \in N \text{ and } v'_i > v_i \text{ for some } i \in N\}.\end{aligned}$$

Trivially, if $v \in \text{PEP}(V)$, then $v \in \partial V \cap \mathbb{R}^n_+$ and $v_i \geq r_i$ for all $i \in N$.

The following proposition is well known. It is essential that V be convex.

Proposition 1. *Let (V, r) be an n-person pure bargaining problem satisfying Assumption 1. For any $\hat{v} \in PEP(V)$, there is a vector of weights $\lambda = (\lambda_1, \ldots, \lambda_n) \geq 0$, $\lambda \neq 0$, such that $\sum_{k \in N} \lambda_k \hat{v}_k \geq \sum_{k \in N} \lambda_k v_k$ for all $v \in V$.*

It is easy to prove Proposition 1 by the supporting hyperplane theorem. Here, we omit the proof. We add the following proposition.

Proposition 2. *A bargaining problem (V, r) satisfies Assumption 1. Let $\hat{v}$ be a solution of the maximization problem:* $\max_v \sum_{k \in N} \lambda_k v_k$ *subject to $v \in V$ and $v_i \geq r_i$ for all $i \in N$. Then, for some $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_n)$, $\hat{v}$ also become a solution of the maximization problem:*

$$\max_v \quad \prod_{k \in N} (v_k - r_k)^{\hat{\theta}_k} \ \text{ subject to } v \in V \text{and } v_i \geq r_i \ \text{ for all } i \in N. \qquad (3)$$

Proof. All that is required is to construct an objective function $\prod_{k \in N}(v_k - r_k)^{\theta_k}$ such that its indifference surface passing through the point $\hat{v}$ comes into contact with the hyperplane $\sum_{k \in N} \lambda_k v_k = \sum_{k \in N} \lambda_k \hat{v}_k$. Choose $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_n)$ satisfying

$$\frac{\lambda_i}{\lambda_j} = \frac{\hat{\theta}_i}{\hat{\theta}_j} \frac{\hat{v}_j - r_j}{\hat{v}_i - r_i}, \ \text{for } i, j \in N, i \neq j, \ \text{and } \sum_{k \in N} \hat{\theta}_k = 1.$$

Then, the function $\prod_{k \in N}(v_k - r_k)^{\hat{\theta}_k}$ would satisfy the required properties. As a result, $\hat{v}$ becomes a solution of the problem (3). □

From Proposition 1 and Proposition 2, it follows that for any $\hat{v} \in \mathrm{PEP}(V)$, there exists a vector $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_n)$ such that $\prod_{k \in N}(\hat{v}_k - r_k)^{\hat{\theta}_k} \geq \prod_{k \in N}(v_k - r_k)^{\hat{\theta}_k}$ for all $v \in V$. Thus, any Pareto-efficient payoff allocation is the Nash bargaining solution of (V, r) with some weight vector $\hat{\theta}$. By Theorem 1, the Nash bargaining solution of (V, r) with the weight vector $\hat{\theta}$ is the limit of SSPE payoff vectors in the bargaining game model $G^{\hat{\theta}}(\rho)$ as $\rho \to 1$.

We then have the following theorem:

Theorem 2. *Let (V, r) be an n-person bargaining problem satisfying Assumption 1. Every Pareto-efficient payoff allocation of (V, r) is realized by the limit of SSPEs of the bargaining game $G^{\hat{\theta}}(\rho)$ as $\rho \to 1$.*

Note that the probability distribution $\hat{\theta}$ to select a proposer in $G^{\hat{\theta}}(\rho)$ is uniquely determined for each Pareto-efficient allocation of (V, r) since V is convex and closed, and the boundary $\partial V \cap \mathbb{R}^n_+$ is nonlevel and smooth.

5 Proof of Theorem 1

The formal proof of Theorem 1 is separated into the following four lemmas: Lemma 1, Lemma 2, Lemma 3, and Lemma 4. First we prove the existence of an SSPE of $G^{\theta}(\rho)$.

Lemma 1. *There exists an SSPE of the game $G^{\theta}(\rho)$ for each $0 \leq \rho < 1$.*

Proof. Let $v = (v_1, \ldots, v_n)$ be the expected payoff vector in an SSPE of $G^{\theta}(\rho)$. Because the feasible set V is convex and v is a convex combination of points in V, therefore, $v \in V$. We denote by v_{-i} and r_{-i} the $(n-1)$-dimensional vector constructed from n-dimensional vectors $v = (v_i)_{i \in N}$ and $r = (r_i)_{i \in N}$ by deleting the i-th coordinate v_i and r_i.

Suppose that player i becomes the proposer at round 1. First, player i can propose a payoff vector $y^i = (y_1^i, \ldots, y_n^i) \in V$. Consider the following problem:

$$\max_{y^i} \; y_i^i \text{ subject to } y_i \geq \rho v_j + (1-\rho) r_j \text{ for } j \neq i, \text{ and } y^i \in V. \qquad (4)$$

Because $v \in V$ and $r \in V$, the convex combination $\rho v + (1-\rho) r \in V$. Therefore, there is at least one vector satisfying the constraints of the above problem. Thus, the constraint set is nonempty. In addition, the constraint set is also bounded, closed and convex since V is convex and closed and $V \cap \mathbb{R}_+^n$ is bounded. Let $g_i^*(\rho v_{-i} + (1-\rho) r_{-i})$ be the maximum value that is attained in the maximization problem. By Berge's maximum theorem, $g_i^*(\rho v_{-i} + (1-\rho) r_{-i})$ is a continuous function.

Second, i might make an unacceptable proposal. In this case, i obtains the expected payoff $\rho v_i + (1-\rho) r_i$.

Let us get the expected payoff to player i. For $v = (v_1, \ldots, v_n)$, we can define a function by

$$\xi_i^\rho(v) = \theta_i g_i^*(\rho v_{-i} + (1-\rho) r_{-i}) + (1-\theta_i)(\rho v_i + (1-\rho) r_i), \qquad (5)$$
$$\text{for} \quad i = 1, \ldots, n.$$

Moreover, let us define a function $\xi^\rho(v) = \prod_{i \in N} \xi_i^\rho(v)$. $\xi^\rho(v)$ is a continuous function from V to itself, and V is a compact and convex set. Then, by Brouwer's fixed point theorem, there exists a fixed point $v^*(\rho) = (v_1^*(\rho), \ldots, v_n^*(\rho)) \in V$ such that for all $i \in N$,

$$v_i^*(\rho) = \xi_i^\rho(v^*(\rho)) = \theta_i g_i^*(\rho v_{-i}^*(\rho) + (1-\rho) r_{-i}) + (1-\theta_i)(\rho v_i^*(\rho) + (1-\rho) r_i).$$

From the fixed point $v^*(\rho)$, we can construct an SSPE of $G^\theta(\rho)$. Consider the strategy combination $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ such that

(i) every player i proposes a solution of the maximization problem (4), and (ii) accepts any proposal y_i if and only if $y_i \geq \rho v_i^*(\rho) + (1-\rho)r_i$.

Because a continuation payoff to every player i is $\rho v_i^*(\rho) + (1-\rho)r_i$, it is easy to check that σ^* prescribes every player's (locally) optimal choice at his every move in $G^\theta(\rho)$. Then, σ^* is an SSPE of $G^\theta(\rho)$ with the expected payoff vector v^*. □

Lemma 2. *In every SSPE σ^* of $G^\theta(\rho)$ with the expected payoff vector v, every player i in N proposes at round 1 a solution y_i^* of the maximization problem:*

$$\max_{y^i} \quad y_i^i \text{ subject to } y_j^i = \rho v_j + (1-\rho)r_j \text{ for } j \neq i, \text{ and } y_i \in V. \qquad (6)$$

The proposal y_i^* is accepted in σ^*. Thus, no delay occurs in equilibrium.

Proof. Because V is convex, we must have $v \in V$. In addition, there is some $y \in V$ such that $y_i > r_i$ for all $i \in N$ since r is contained in the interior of V. Since $v \in V$, $y \in V$, and V is convex, it holds that $\rho v + (1-\rho)y \in V$ for any $0 \leq \rho < 1$. Because $y_j > r_j$ for all $j \in N$, we have

$$\rho v_j + (1-\rho)y_j > \rho v_j + (1-\rho)r_j \quad \text{for all } j \in N.$$

Thus, there exists $\hat{y} = (\hat{y}_j)_{j \in N} \in V$ such that

$$\hat{y}_j > \rho v_j + (1-\rho)r_j \quad \text{for all } j \in N. \qquad (7)$$

We note that the expected payoff to player j is $\rho v_j + (1-\rho)r_j$ when player j as a responder rejects the proposal. Therefore, at a solution of the problem (6), proposer i obtains the maximum payoff among the proposals that will be accepted. In addition, any unacceptable proposal yields to player i at most $\rho v_i + (1-\rho)r_i$. By (7), the payoff to proposer i at a solution of the problem

(6) is strictly greater than $\rho v_i + (1-\rho)r_i$. Hence, player i will propose a solution of the problem (6) at round 1.

Since any proposal y_j is accepted by every player $j \in N$ if and only if $y_j \geq \rho v_j + (1-\rho)r_j$, then a solution of the problem (6) will be accepted. $\square$

Lemma 3. *For a $v(\rho) = (v_1(\rho), \ldots, v_n(\rho))$, there exists an SSPE of $G^\theta(\rho)$ with the expected payoff vector $v(\rho)$ if and only if the expected payoff vector $v(\rho)$ satisfies, for $i = 1, \ldots, n$,*

$$v_i(\rho) = \theta_i g_i^*(\rho v_{-i}(\rho) + (1-\rho)r_{-i}) + (1-\theta_i)(\rho v_i(\rho) + (1-\rho)r_i), \quad (8)$$

where $g_i^*(\cdot)$ is the maximum value of the problem:

$$\max_y \; y_i \; \text{ subject to } y_j \geq \rho v_j(\rho) + (1-\rho)r_j, j \neq i, j \in N, \text{and } y \in V.$$

Proof. (if) We will construct an SSPE of $G^\theta(\rho)$ with the expected payoff vector $v(\rho)$ satisfying (8). Define the strategy combination σ such that at each round t, every player i proposes $g_i^*(\rho v_{-i}(\rho) + (1-\rho)r_{-i})$, and accepts any proposal y_i if and only if $y_i \geq \rho v_i(\rho) + (1-\rho)r_i$. As in the proof of Lemma 1, it is easy to see that σ prescribes every player's optimal choice.

(only-if) It follows from Lemma 2 that every player i gets the payoff $g_i^*(\rho v_{-i}(\rho) + (1-\rho)r_{-i})$ if i is the proposer and has $\rho v_i(\rho) + (1-\rho)r_i$ if i is the responder. Recall that player i is selected as a proposer with probability θ_i and becomes a responder with probability $1-\theta_i$. In addition, from Lemma 2, the proposal is accepted at round 1. Therefore, by definition of the game $G^\theta(\rho)$, we can obtain the equation (8). $\square$

Lemma 4. *Let $v^*(\rho) = (v_1^*(\rho), \ldots, v_n^*(\rho))$ be an SSPE payoff vector in $G^\theta(\rho)$ for each ρ and v^* is a limit point of $v^*(\rho)$ as $\rho \to 1$. Then, v^* is the asymmetric Nash bargaining solution of (V, r) with the weight vector θ.*

Proof. Let $x_i(\rho)$ denote the payoff $g_i^*(\rho v_{-i}^*(\rho) + (1-\rho)r_{-i})$ that every player i obtains in an SSPE of $G^\theta(\rho)$ if i is the proposer, i.e., $x_i(\rho) = g_i^*(\rho v_{-i}^*(\rho) + (1-\rho)r_{-i})$ for all $i \in N$. By Lemma 2 and Lemma 3, we have that for every $i \in N$

$$H(\rho v_1^*(\rho) + (1-\rho)r_1, \ldots, x_i(\rho), \ldots, \rho v_n^*(\rho) + (1-\rho)r_n) = 0. \tag{9}$$

Let us define the vector

$$z^i(\rho) = (\rho v_1^*(\rho) + (1-\rho)r_1, \ldots, x_i(\rho), \ldots, \rho v_n^*(\rho) + (1-\rho)r_n).$$

$z^i(\rho)$ represents the payoff vector proposed by player i at round 1 in an SSPE of $G^\theta(\rho)$.

By Lemma 3, we can obtain

$$v_i^*(\rho) = \theta_i x_i(\rho) + (1-\theta_i)(\rho v_i^*(\rho) + (1-\rho)r_i), \text{ for } i = 1, 2, \ldots, n. \tag{10}$$

Rearranging (10), we get

$$x_i(\rho) = \frac{1-\rho}{\theta_i} v_i^*(\rho) + \rho v_i^*(\rho) - \frac{1-\theta_i}{\theta_i}(1-\rho)r_i, \text{ for } i = 1, \ldots, n. \tag{11}$$

Since $\lim_{\rho\to 1} v^*(\rho) = v^*$ and $x_i(\rho)$ is represented as (11), we have that $\lim_{\rho\to 1} x_i(\rho) = v_i^*$ for all $i \in N$. This implies that the proposals by all players in an SSPE converge to the same payoff vector. Thus,

$$\lim_{\rho\to 1} z^1(\rho) = \lim_{\rho\to 1} z^2(\rho) = \cdots = \lim_{\rho\to 1} z^n(\rho) = v^*. \tag{12}$$

For any $i, j \in N$, $i \neq j$, it follows from (9) that $H(z^i(\rho)) - H(z^j(\rho)) = 0$. By Taylor's theorem, there exists some $0 < t < 1$ such that

$$\begin{aligned} &H(z^i(\rho)) - H(z^j(\rho)) \\ &= \frac{v_i^*(\rho) - r_i}{\theta_i} \frac{\partial H}{\partial x_i}(tz^i(\rho) + (1-t)x^j(\rho)) \\ &\qquad - \frac{v_j^*(\rho) - r_j}{\theta_j} \frac{\partial H}{\partial x_j}(tz^i(\rho) + (1-t)z^j(\rho)) = 0. \end{aligned} \tag{13}$$

By (12) and (13), we can obtain, as $\rho \to 1$,

$$\frac{v_i^* - r_i}{\theta_i}\frac{\partial H}{\partial x_i}(v^*) = \frac{v_j^* - r_j}{\theta_j}\frac{\partial H}{\partial x_j}(v^*), \quad i, j \in N, i \neq j, \tag{14}$$

$$H(v^*) = 0. \tag{15}$$

Thus, v^* satisfies the Kuhn-Tucker condition (1), (2) of the maximization problem for the Nash bargaining solution of (V, r) with the weight vector θ. □

By combining Lemma 1 with Lemma 4, we can obtain Theorem 1.

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