A Role of Redistribution for the Provision of Public Good in the Social Contractual State

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Abstract

We present an n-person noncooperative game model of public good provision. This economy consists of a high-income and a low-income group. Our game is a state formation game that constitutes a punishment rule and installs an enforcer for the provision of public goods. We show that a voluntary redistribution by the high-income group makes it possible to provide the public good. In such an equilibrium, the real tax burden of a high-income individual is heavier than that of a low-income individual, thus, progressive income taxes emerge.

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1 Introduction

In this paper, we examine the relationships between income redistributions, the progressivity of taxation, and the provision of public good. A public good usually becomes an under-provision because of the '*free-rider problem*'. We show that income redistributions from the high-income individuals to the low-income ones make it possible to increase the provision of public good.

We consider an extended model of Okada and Skakibara (1991) and Okada (1993) to a two-class economy, which consists of high-income individuals and low-income individuals. It is assumed that a marginal utility of income for the high-income individual is smaller than that for the low-income individual and a marginal utility of public good is common at each initial endowments level. The following situations are considered. All individuals choose to contribute an equal amount of private good to the provision of the public good or not. No public good is provided because the non-contribution is a dominant strategy for every individual. Okada and Sakakibara presented the game of institutional arrangements. In this game, there is a preplay negotiation stage in which all individuals negotiate for a punishment rule against the non-contributors and for installing an enforcement agency. These rules are decided by the unanimity rule. They showed that in the economy with identical individuals, the public good can be provided under an institution with a punishment rule. They called the institution with an enforcement agency a *social contractual state*. Since we extend the model to a two-class economy, a situation would arise such that the public good is not provided by any institutional arrangements for a punishment. We show that a voluntary income-redistribution by the high-income individuals induce to the cooperative provision of public good under the institution arrangement game. The equilibrium outcome is Pareto-superior to the outcome under no public good. Moreover, a tax burden on the high-income individual is heavier than that on the low-income individual; the structure of real tax burdens is progressive in income. Our results indicate the possibility of the transition from the equal share tax to the progressive tax. Income redistributions would be made by the high-income individuals for promoting the provision of public good.

Economists have been provided several rationales for redistribution[1]. One of the most famous rationales is seen in *utilitarianism*. The celebrated utilitarian, Pigou (1932) asserted that a redistribution of income from high-income individuals to low-income individuals increased the level of social welfare if all individuals have identical utility function and have no disincentive effect on labor supply. The social welfare represents social justice that transcends individual decision-making, and income redistribution is justified as a compulsory policy instrument. Another rationale has been based on the *altruism* of high-income individuals, which was firstly presented by Hochman and Rogers (1969) and Thurow (1971). Under this approach, each high-income individual is envisaged as gaining some satisfaction from the well-being of the low-income individuals. In formal terms, the utility of a high-income individual depends positively on the utilities of low-income individuals. Then, high-income individuals voluntarily make income redistribution and the Pareto-improvement is realized through the income redistribution. Moreover, Varian (1981) has rationalized redistribution as a *social insurance* under the uncertainty about individual's income. In this paper, we show that high-income individuals voluntarily redistribute their income in order to induce the cooperative provision of public good. The redistribution is that for the sake of the provision of public good. In our model, there is no value judgment that transcends the determinants of individually rational behavior and no individual has an altruistic preference. In addition, there is no uncertainty about incomes. Therefore, the redistribution for the cooperative provision of public good is a new rationale.

Let us refer to the relation between our result and the *neutrality theorem* in the context of the private provision of public goods by Bergstrom, Blume and Varian (1986) and Warr (1983). We will show that income redistribution can lead to the increase of the public good provision. This result seems to be inconsistent with the neutrality theorem. But, preplay communications have not been considered in the context of the private provision of public

[1]Boadway and Keen (2000) give a detailed and comprehensive survey of the research in income redistribution.

good. Therefore, our result is not directly a counterexample of the neutrality theorem. In addition, our result crucially depend on assuming a binary choice set with no contribution and one unit contribution to the public good for every individual. In the literatures of the private provision of public goods, each individual can choose any level of contribution of the private good.

This paper is organized as follows. Section 2 presents a basic framework. Section 2.1 establishes the model of a public good economy with high-income individuals and low-income individuals. Section 2.2 presents the formal model of the state formation game and defines a noncooperative solution concept for it. Then, we show that the public good is not provided under any enforcement system without income redistribution. Section 3 allows an income redistribution between high-income individuals and low-income individuals and shows that the public good may be provided in a noncooperative solution of our state formation game. Moreover, we give a sufficient condition for a public good to be provided.

2 Framework

2.1 Basic Model

Let $N = \{1, 2, \ldots, n\}$ be the set of players. The set N is divided into two income groups, a high-income group N_1 and a low-income group N_2. The number of N_1 is n_1, and that of N_2 is $n_2 = n - n_1$.

This economy consists of two goods; a public good and a private good. A high-income individual $i \in N_1$ is endowed with y_1 units of the private good, and a low-income one $j \in N_2$ is endowed with y_2, where $y_1 > y_2$. Each private good endowment represents his/her income level.

Assume that one unit of the private good can be transformed into one unit of the public good. Therefore, the amount of the public good provision is equal to the sum of individuals' contributions of the private good to a state (government). A public good is consumed in equal amounts by all individuals. The utility function of individual $i \in N_1$ and $j \in N_2$ is given by[2]

$$u_{1i}(G, c_i) \overset{\text{def}}{=} G + \beta(y_1)c_i, \quad u_{2j}(G, c_j) \overset{\text{def}}{=} G + \beta(y_2)c_j, \tag{1}$$

where G is the consumption of public good and c_i (c_j) is the consumption of private good for individual i (j). $\beta(\cdot)$ represents a marginal utility of the private good.

Assumption 1. The function $\beta(\cdot)$ is strictly decreasing with income y.

By Assumption 1 and $y_1 > y_2$, $\beta(y_1) < \beta(y_2)$, i.e., the marginal benefit of private good for the high-income individual is smaller than that for the low-income individual. In this paper, we consider the question of whether every individual contributes one unit of private good to the provision of the public good or not. Formally every individual has two potential actions a_i: C (cooperation) and D (defection). The action C represents a contribution of one unit private good toward the public good, and D represents the decision not to contribute. The value $\beta(y_i)$ denotes a marginal rate of substitution of a private good for a public good at (y_i, G). If a decreasing law of the marginal income utility is satisfied and a marginal utility of public good is invariable, the marginal rate of substitution for the high-income individual is smaller than that for the low-income individual. Equivalently, the net benefit from the provision of public good with one-unit loss of the private-good consumption for a high-income individual is greater than that for a low-income individual.

For $a = (a_1, \ldots, a_n)$, the payoff for individual $i \in N_1$ and $j \in N_2$ is given by, respectively,

$$u_{1i} = \begin{cases} (h_i + 1) + \beta(y_1)(y_1 - 1) & \text{if } \ a_i = C, \\ h_i + \beta(y_1)y_1 & \text{if } \ a_i = D. \end{cases}$$

$$u_{2j} = \begin{cases} (h_j + 1) + \beta(y_2)(y_2 - 1) & \text{if } \ a_j = C, \\ h_j + \beta(y_2)y_2 & \text{if } \ a_j = D, \end{cases}$$

[2]The symbol '$\overset{\text{def}}{=}$' means that the left hand side is defined by the right hand side.

where h_i (h_j) is the number of all individuals except i (j) to select cooperation.

We add the following assumptions of β.

Assumption 2. (i) $1 < \beta(y_1),\ \ 1 < \beta(y_2)$, (ii) $n_1 < \beta(y_1) < n$, (iii) $n < \beta(y_2)$.

Condition (i) in Assumption 2 means that $(h+1)+\beta(y)(y-1) < h+\beta(y)y$. It implies that defection dominates cooperation for every individual. Thus the action combination $(D, \dots, D)$ of individuals is a unique noncooperative (Nash) equilibrium of this game, and no public good is provided[3]. In other words, the *'free-rider problem'* always arises when everyone acts selfishly. We call this equilibrium $(D, \dots, D)$ the *'anarchic state of nature'*.

Next, condition (ii) means that $n + \beta(y_1)(y_1 - 1) > \beta(y_1)y_1$ and $n_1 + \beta(y_1)(y_1 - 1) < \beta(y_1)y_1$. It implies that the payoff for a high-income individual $i \in N_1$ in the case that all individuals cooperate is greater than the payoff in the case that no individual cooperates, i.e., the case of the anarchic state of nature. Still, the payoff for $i \in N_1$ in the case that only high-income individuals cooperate is smaller than his/her payoff in the anarchic state of nature.

From condition (iii), $n + \beta(y_2)(y_2 - 1) < \beta(y_2)y_2$; the payoff for a low-income individual $j \in N_2$ in the case that all individuals cooperate is smaller than that in the anarchic state of nature.

Let point out a difference between our study and that of Okada and Sakakibara (1991). Okada and Sakakibara assumed that the society consists of identical individuals and that $1 < \beta < n$. In their situation, every individual is better off by selecting defection than by selecting cooperation, regardless of what all other individuals select. Thus, defection is the individually rational choice. At same time, the outcome that arises if all individuals select cooperation dominates the outcome with defection of all individuals in the sense of Pareto efficiency. Cooperation is the socially rational choice. Their

[3]No contribution of each individual only represents less contributions than the cooperative action C. We can easily modify the model in which all individuals contribute to the voluntary provision of public good in the anarchic state of nature.

situation is essentially an n-person prisoners' dilemma. In our situation, the payoff for a low-income individual under cooperation of all individuals is smaller than the payoff in the case that all individuals select defection. On the other hand, a high-income individual faces the same situation as that in Okada and Sakakibara. Therefore, cooperation is desirable for high-income individuals, but is undesirable for low-income individuals in our model.

2.2 A State Formation Game

In this economy, defection (D) dominates cooperation (C) for every individual, and the action combination $(D, \cdots, D)$ is a unique noncooperative equilibrium of the game without an enforcement system. Thus, no public good is provided in the anarchic state of nature. Okada and Sakakibara (1991) and Okada (1993) have presented a noncooperative game model of institutional arrangements in which individuals negotiate for creating the enforcement agency and for constituting a punishment rule against defections[4]. They showed that cooperation by all individuals is reached under a certain kind of enforcement system. In our situation, however, the public good will not be provided by the same enforcement system unless there exists a redistribution of income.

Let us explain our noncooperative game shortly. All individuals negotiate for installing an enforcement agency and also for constituting a punishment rule against an action of defection. An agreement for installing an enforcement agency and for setting the amount of punishment can be reached under the unanimity rule of collective choice. If the enforcement agency is installed, the agency imposes punishment on the member of the society who selects defection. In addition, the enforcement agency collects tax and produces a public good. Then, All individuals in the society decide independently whether they should cooperate or not.

[4]The game of Okada and Sakakibara (1991) had a 'Participation Decision Stage'. In this stage, all players could decide whether or not to participate in bargaining for installing an enforcement system. In our game, participation in such bargaining is compulsory for all individuals.

In our game, the action of cooperation is one unit contribution of a private good. Therefore, cooperation can be regarded as a tax payment for a public good provision and defection can be interpreted as a behavior of tax evasion. Our game describes the bargaining to form an organization that collects tax, provides a public good, and punishes the tax evader. This organization can be regarded as a state. So, we call our noncooperative game the *state formation game*.

Formally, the state formation game consists of the two stages as follows.

Stage 1: The bargaining stage for an enforcement system

In this stage, all individuals negotiate for installing an enforcement agency and also for the punishment rule. Every individual $k \in N$ in the society simultaneously selects a nonnegative real number q_k. The number q_k is the amount of punishment which the enforcement agency will impose on deviators from the contribution to the provision of a public good. For a decision vector $(q_k : k \in N)$, the punishment p of the enforcement agency is determined by the unanimity rule:

$$p = \begin{cases} q & \text{if } \; q = q_k \text{ for all } \; k \in N, \\ 0 & \text{otherwise.} \end{cases}$$

An enforcement agency is installed if and only if all members in N can reach a unanimous agreement on the amount of punishment. We assume that the enforcement agency monitors only whether each individual contributes one unit of private good and cannot observe the deviators' income level. It follows that every high- and low-income individual faces the same amount of punishment against defection[5]. Furthermore, we do not consider the costs of monitoring and of punishments.

Stage 2: The action decision stage

Given p, all players decide independently whether they contribute one unit of the private good to provide the public good or not. Every player $k \in N$ simultaneously selects his/her action $a_k = C$ or D. For an action vector

[5]Even if a different punishment between a high- and a low-income individual is allowed, the following main results are not changed.

$a = (a_1, \dots, a_n)$, the final payoff $F_i(a)$ for $i \in N_1$ and $F_j(a)$ for $j \in N_2$ are determined as follows.

$$F_i(a) = \begin{cases} (h_i(a)+1) + \beta(y_1)(y_1 - 1), & a_i = C, \\ h_i(a) + \beta(y_1)(y_1 - p), & a_i = D, \end{cases} \tag{2}$$

$$F_j(a) = \begin{cases} (h_j(a)+1) + \beta(y_2)(y_2 - 1), & a_j = C, \\ h_j(a) + \beta(y_2)(y_2 - p), & a_j = D, \end{cases} \tag{3}$$

where $h_i(a)(h_j(a))$ is the number of contributors except player $i \in N_1 (j \in N_2)$ in the action vector a.

The state formation game described above is formally represented by an extensive form game, denoted by Γ. We assume that every individual makes his/her decision with perfect information of the previous stages at every stage of the game Γ and that the rule of the game is a common knowledge.

A behavior strategy σ_i for player i in the game Γ is a function that assigns a randomized choice to each of his/her decision nodes in Γ, and $\sigma = (\sigma_1, \dots, \sigma_n)$ is a behavior strategy combination.

Let us define our noncooperative solution concept of the game Γ.

Definition 1. A behavior strategy combination $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ of the game Γ is said to be a *noncooperative solution* of Γ if and only if it satisfies the following conditions.

(1) (subgame perfectness) σ^* is a subgame perfect equilibrium point of Γ.

(2) (symmetry-invariance) In every σ^*-stage game of Γ, σ^* induces a symmetry-invariant equilibrium point, i.e., an equilibrium point that is independent of permutations of players and of their choices.

(3) (payoff-dominance) σ^* induces an equilibrium point on every σ^*-stage game of Γ which payoff-dominates all other symmetry-invariant equilibrium points of the σ^*-stage game.

For a detail definition of symmetry-invariant and payoff-dominant equilibrium points, see Harsanyi and Selten (1988). The computation of our noncooperative solution of Γ can be done by the usual *backward induction procedure* in the theory of extensive games.

A noncooperative solution of the state formation game is summarized as follows.

Proposition 1. *In a noncooperative solution of the state formation game, where no income redistribution between high- and low-income individuals is allowed, 'the enforcement agency is not installed and no agreement on the punishment rule' or 'an agreement on the punishment $p = 0$' is achieved in the bargaining stage for an enforcement system. Moreover, in the action stage, no individual contributes the one unit of private goods to the provision of public good on the equilibrium path. Thus, the anarchic state of nature $(D, \ldots, D)$ is realized. A unique solution payoff vector $(F_k)_{k \in N}$ is $F_i = \beta(y_1)y_1$ for all $i \in N_1$ and $F_j = \beta(y_2)y_2$ for all $j \in N_2$.*

The proof is ommited. In the above state formation game, every low-income individual could not better off by cooperation than deviation how many individuals would cooperate. Then, all low-income individuals do not contribute. Every high-income individual does not contribute when all low-income individuals deviate. Therefore, the public good does not be provided, i.e., a state does not be formed. If income redistribution between high- and low-income individuals is allowed, the situation dramatically change. This case will be investigated in the next section.

3 Cooperation by Redistribution

3.1 Voluntary Redistribution

We show that voluntary income redistributions by high-income individuals give rise to the cooperative provision of the public good in a noncooperative solution.

We add the income redistribution stage to the two-stage game in Section 2.2. The bargaining stage for the enforcement system follows the income redistribution stage, and then the action decision stage game is played.

Stage 0: Income redistribution stage

In this stage, all high-income individuals $i \in N_1$ redistribute a part of their income to low-income individuals $j \in N_2$. Let t be the amount of the income redistribution per high-income person. We assume that each high-income individual redistributes the same amount of income t and that the aggregate amount tn_1 is divided equally among all low-income individuals. Therefore, the amount of income received by a low-income individual is tn_1/n_2. The amount t is determined by a take-it-or-leave-it offer from high-income individuals.

It is true that there exist other bargaining procedures to determine t, but we adopt the bargaining procedure based on a take-it-or-leave-it offer in order to emphasize the aspect of voluntary redistribution by the high-income individuals. In addition, this procedure simplifies our model.

Let us restrict the range of redistribution as follows.

Assumption 3. The amount of income redistribution by each high-income individual t is restricted to the interval $[0, (y_1 - y_2)n_2/n]$, i.e., the redistribution t satisfies $y_1 - t \geq y_2 + tn_1/n_2$.

Assumption 3 excludes the case that the income of a high-income individual $i \in N_1$ is lowered below that of a low-income individual $j \in N_2$ by the income redistribution. Thus, redistribution does not change the income order of group $i \in N_1$ and $j \in N_2$. The order preserving of incomes is one of the natural axioms of the tax system; see Young (1988).

After the income redistribution stage, the bargaining stage for an enforcement system and the action decision stage are played. Our game consists of a three-stage game. We denote an extensive form game in this section by $\hat{\Gamma}$.

3.2 Characterization of Noncooperative Solution

We characterize our noncooperative solution of $\hat{\Gamma}$. A noncooperative solution can be obtained by the backward induction procedure. Let start with the action decision stage game $\hat{G}^2$ of $\hat{\Gamma}$. Since the income redistribution stage is added, the action decision stage game depends on punishment p, population structure (N, N_1, N_2) and also an income redistribution level t. We denote the action decision stage game by $\hat{G}^2(N, N_1, N_2, p, t)$. In this stage game, every individual has two pure strategies, C and D, and the payoff function $F_i(a), F_j(a)$ for $i \in N_1$ and $j \in N_2$ are given by

$$F_i(a) = \begin{cases} (h_i(a) + 1) + \beta(y_1 - t)(y_1 - t - 1), & a_i = C, \\ h_i(a) + \beta(y_1 - t)y_1 - t - p, & a_i = D, \end{cases} \tag{4}$$

$$F_j(a) = \begin{cases} (h_j(a) + 1) + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1), & a_j = C, \\ h_j(a) + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - p), & a_j = D. \end{cases} \tag{5}$$

These payoff functions are obtained by substituting $y_1 - t$ and $y_2 + tn_1/n_2$ into y_1 and y_2 in (2), (3). Then, we obtain the following lemma about the action decision stage game.

Lemma 2. *The action decision stage $\hat{G}^2(N, N_1, N_2, p, t)$ has a unique solution $a = (a_k)_{k \in N}$ such that*

(i) *If $0 \leq p < 1 - 1/\beta(y_1 - t)$, then $a_k = D$ for all $k \in N$.*

(ii) *If $1 - 1/\beta(y_1 - t) \leq p < 1 - 1/\beta(y_2 + tn_1/n_2)$, then $a_i = C$ for all $i \in N_1$ and $a_j = D$ for all $j \in N_2$.*

(iii) *If $p \geq 1 - 1/\beta(y_2 + tn_1/n_2)$, then $a_k = C$ for all $k \in N$.*

Next the bargaining stage game for an enforcement system is considered. We denote the bargaining stage game for an enforcement system by $\hat{G}^1(N, N_1, N_2, t)$. The game $\hat{G}^1(N, N_1, N_2, t)$ can be described as a game in which every individual $k \in N$ selects a punishment level q_k. For a strategy combination $q = (q_k)_{k \in N}$, the payoff for $i \in N_1$ and $j \in N_2$ is given by taking account of Lemma 2 as follows.

(a) If $q_k = p$ for all $k \in N$ and $p \geq 1 - 1/\beta(y_2 + tn_1/n_2)$, then

$$F_i(q) = n + \beta(y_1 - t)(y_1 - t - 1), \quad i \in N_1, \tag{6}$$

$$F_j(q) = n + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1), \quad j \in N_2, \tag{7}$$

(b) If $q_k = p$ for all $k \in N$ and $1 - 1/\beta(y_1 - t) \leq p < 1 - 1/\beta(y_2 + tn_1/n_2)$, then

$$F_i(q) = n_1 + \beta(y_1 - t)(y_1 - t - 1), \quad i \in N_1, \tag{8}$$

$$F_j(q) = n_1 + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - p), \quad j \in N_2, \tag{9}$$

(c) If $q_k = p$ for all $k \in N$ and $0 \leq p < 1 - 1/\beta(y_1 - t)$, then

$$F_i(q) = \beta(y_1 - t)(y_1 - t - p), \quad i \in N_1, \tag{10}$$

$$F_j(q) = \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - p), \quad j \in N_2, \tag{11}$$

(d) If no agreement on the punishment level is reached, i.e., there exist $i,\ j \in N$ such that $q_i \neq q_j$ and $i \neq j$, then

$$F_i(q) = \beta(y_1 - t)(y_1 - t), \quad i \in N_1, \tag{12}$$

$$F_j(q) = \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2), \quad j \in N_2. \tag{13}$$

By comparing the above payoff functions each other, we can obtain the following lemma.

Lemma 3. (i) *If a triplet (t, n_1, n_2) satisfies the following conditions;*

$$n + \beta(y_1 - t)(y_1 - t - 1) > \beta(y_1 - t)(y_1 - t), \text{ and}$$
$$n + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1) > \beta(y_2 + tn_1/n_2)(y_2 + tn_2/n_1),$$

where $n = n_1 + n_2$, then an agreement on punishment $p \geq 1 - 1/\beta(y_2 + tn_1/n_2)$ is reached in the bargaining stage for an enforcement system. Moreover, the bargaining stage game $\hat{G}^1(N, N_1, N_2, t)$ for an enforcement system has an unique solution payoff vector $(F_k)_{k \in N}$ such that $F_i = n + \beta(y_1 - t)(y_1 - t - 1)$ for all $i \in N_1$ and $F_j = n + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1)$ for all $j \in N_2$.

(ii) *If a triplet (t, n_1, n_2) does not satisfy the above conditions, either 'no agreement on the punishment rule' or 'an agreement on $p = 0$' is reached in the bargaining stage for an enforcement system. Then, the bargaining stage game $\hat{G}^1(N, N_1, N_2)$ for an enforcement system has a unique solution payoff vector $(F_k)_{k \in N}$ such that $F_i = \beta(y_1 - t)(y_1 - t)$ for all $i \in N_1$ and $F_j = \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2)$ for all $j \in N_2$.*

Note that there is a case in which all individuals agree on a punishment level p such that all individuals select cooperation. It is impossible to agree on such a punishment level in a state formation game without income redistribution. Thus, the addition of the income redistribution stage affects the possibility of cooperation in our state formation game.

Finally, the income redistribution stage game is analyzed. We denote the income redistribution stage game by $\hat{G}^0(N, N_1, N_2)$. The strategies of players in this stage game $\hat{G}^0(N, N_1, N_2)$ consist of proposals for an income redistribution t by every high-income individual and responses to the proposal by every low-income individual. Thus, every high-income individual make some proposal for an income redistribution level in the first move, and every low-income individual either accepts or rejects the proposal in the second move. It is clear that all low-income individuals accept the proposal satisfying $t \geq 0$ as their optimal responses. On the other hand, the strategy of a high-income individual depends on the population structure (N, N_1, N_2). Since we assume a take-it-or-leave-it offer by the high-income individuals, we only consider whether there exists a positive t such that all high-income individual are better off. Then we can characterize the noncooperative solution of our state formation game with income redistribution as follows.

Proposition 4. (i) *If there exists* $t \in [0, (y_1 - y_2)n_2/n]$ *such that the following conditions are satisfied;*

$$n + \beta(y_1 - t)(y_1 - t - 1) \geq \beta(y_1)y_1, \text{ and} \tag{14}$$

$$n + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1) \geq \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2), \tag{15}$$

then, in the income redistribution stage, every high-income individual offers the minimum t^* *which satisfies (14) and (15), and every low-income individual accepts his/her offer. Moreover, the income redistribution stage game* $\hat{G}^0(N, N_1, N_2)$ *has a unique solution payoff vector* $(F_k)_{k \in N}$ *such that* $F_i = n + \beta(y_1 - t^*)(y_1 - t^* - 1)$ *for all* $i \in N_1$ *and* $F_j = n + \beta(y_2 + t^* n_1/n_2)(y_2 + t^* n_1/n_2 - 1)$ *for all* $j \in N_2$.

(ii) *If there is no* $t \in [0, (y_1 - y_2)n_2/n]$ *satisfying (14) and (15), every high-income individual selects* $t = 0$ *as an equilibrium strategy. Then the* *income redistribution game* $\hat{G}^0(N, N_1, N_2)$ *has a unique solution payoff vector* $(F_k)_{k\in N}$ *such that* $F_i = \beta(y_1)y_1$ *for all* $i \in N_1$ *and* $F_j = \beta(y_2)y_2$ *for all* $\in N_2$*.*

Proposition 4 means that, if there exists t that satisfies (14) and (15), each individual performs the following action in our noncooperative solution. In the redistribution stage, all high-income individuals offer an income redistribution t^*, and all low-income individuals accept their offer. Then, income redistribution occurs in the society. In the bargaining stage for an enforcement system, all individuals agree to install an enforcement agency and to constitute the punishment rule $p \geq 1 - 1/\beta(y_2 - tn_1/n_2)$. In the action decision stage, every individual selects cooperation C, i.e., contributes one unit of private good as a tax payment. As a result, the public good is provided.

When income redistribution was prohibited in Section 2.2, the public good could not be provided in this society. Thus, Proposition 4 shows that income redistribution plays an important role for providing the public good cooperatively. In addition, we should note that a high-income individual contributes one unit of private good plus redistribution t^* and a low-income individual pays one unit of private good minus t^*n_1/n_2 if the public good is provided. Thus, the structure of real tax burdens is progressive. From a comparison of Proposition 4 with Proposition 1 it seems reasonable to say that a public good is more easily provided under a progressive income tax rather than a head tax; an equal tax-burden to all individuals.

Moreover, the solution payoffs for a high-income individual $i \in N_1$ and a low-income individual $j \in N_2$ when the public good is provided satisfy the following inequalities,

$$n + \beta(y_1 - t^*)(y_1 - t^* - 1) \geq \beta(y_1)y_1,$$
$$n + \beta(y_2 + t^*n_1/n_2)(y_2 + t^*n_1/n_2 - 1) > \beta(y_2)y_2,$$

where y_1 and y_2 are payoffs for a high-income and low-income individuals in the anarchic state of nature. This means that the equilibrium outcome is Pareto-superior to the outcome in the anarchic state of nature. Thus, income redistribution induces the Pareto-improvement in our model.

3.3 Sufficient Conditions for Cooperation

It is clear from Proposition 4 that the public good is provided only if there exists t which satisfies the inequalities (14) and (15). In this section, we will investigate under what situations the public good is provided. In our state formation game, the provision of the public good in a noncooperative solution necessarily involves income redistribution from high-income individuals to low-income individuals. Therefore, by presenting a condition for a public good to be provided, we also characterize under what situations high-income individuals voluntarily redistribute their income. It, however, is difficult for us to clarify the situations by using the sufficient conditions (14) and (15) for the public good to be provided. Thus, we prepare a simple condition for the public good to be provided, i.e.,

$$n + \beta(\bar{y})(\bar{y} - 1) > \beta(y_1)y_1, \tag{16}$$

where $\bar{y} \stackrel{\text{def}}{=} (y_1 n_1 + y_2 n_2)/n$ denotes the average income of all individuals in this society. The situation satisfying condition (16) is interpreted as the situation that (14) and (15) are satisfied at $t = (y_1 - y_2)n_2/n$. Therefore, it follows from Proposition 4 that the public good is necessarily provided in a noncooperative solution of our game if condition (16) is satisfied.

We can obtain some implications about income redistribution from condition (16). First, the left hand side of (16) is an increasing function with the average income of all individuals. This implies that the income redistribution for a cooperative provision of the public good will occur in a society with a high average income. Second, given $n = n_1 + n_2$ and y_1, where $y_1 > y_2$, the increase of y_2 causes the increase of the average income. Thus, the left hand side of (16) increases with the increase of y_2. and y_1, the right hand side, is constant. This means that the voluntary income redistribution for a public good provision occurs between individuals with small income differences.

There are some emprical studies which have examined the links between pre-tax income inequality and redistribution. Redistribution was measured by either the share of transfer-payments in GDP, average and marginal tax rates, or education expenditures. Lindert (1996) used as a measure of redistribution the shares of transfer-payments for social security, welfare, unemployment, health, and education in GDP. Perotti (1994) used the ratio of total transfers to GDP. Both Lindert and Perotti have pointed out a negative correlation between income inequality and redistribution. Our implication is consistent with their empirical evidences. Bénabou (1997), Peltzman (1980) and Persson (1995) have also paid attention to recurring signs that redistribution tends to be greater the less underlying pre-tax inequality there is. This effect has been firstly noted by Peltzman. Bénabou explained the effect in the simple growth model, and Peltzman and Persson emphasized the political aspects between income groups. We give another explanation in the process of a state formation.

Appendix

Proof of Lemma 2: The payoff functions for individuals $i \in N_1$, $j \in N_2$ in $\hat{G}^2(N, N_1, N_2, p)$ are given by (4) and (5). From Assumption 2 and 3, every high-income individual $i \in N_1$ has the dominant strategy D (or C) if $0 \leq p < 1 - 1/\beta(y_1 - t)$ ($p > 1 - 1/\beta(y_1 - t)$). Similarly, we can see that every low-income individual $j \in N_2$ has the dominant strategy D (or C) if $0 \leq p < 1 - 1/\beta(y_2 + tn_1/n_2)$ ($p > 1 - 1/\beta(y_2 + tn_1/n_2)$) respectively. Therefore, we will prove enough that the lemma is true in the two cases that $p = 1 - 1/\beta(y_1 - t)$ and $p = 1 - 1/\beta(y_2 + tn_1/n_2)$.

When $p = 1 - 1/\beta(y_1 - t)$, we can easily see that the two strategies C and D are indifferent for a high-income individual. Therefore, any mixed strategy combination between C and D is a Nash equilibrium point. Let $b^* = (b_1^*, \ldots, b_n^*)$ be the symmetry-invariant equilibrium point with $b_i^* = C$ for all $i \in N_1$ and $b_j^* = D$ for all $j \in N_2$. We shall show that b^* payoff-dominates any other symmetric equilibrium point $b = (b_1, \ldots, b_n)$. Note that D is the dominant strategy for $j \in N_2$, i.e., $b_j^* = b_j = D$ for all $j \in N_2$. For every $T \subset N_1$ and every pure strategy $a_i = C$, or D, we denote by $b_T(a)$ the probability that all player in T select the pure strategy a in the equilibrium point b. Thus, $b_T(a) = \prod_{i \in T} b_i(a)$ where $b_i(a)$ is the probability that the mixed strategy b_i assigns to a. Since $b \neq b^*$, there exists some $T \subset N_1$ such that $b_T(D) > 0$. Let $F_i(b)$ be the expected payoff function for player i for a mixed strategy combination b. Then, for all $i \in N_1$,

$$
\begin{aligned}
&F_i(b) \\
&= \sum_{i \in T \subset N_1} b_T(C) b_{N_1 - T}(D)[|T| + \beta(y_1 - t)(y_1 - t - 1)] \\
&\qquad + \sum_{i \notin T \subset N_1} b_T(C) b_{N_1 - T}(D)[|T| + \beta(y_1 - t)(y_1 - t - p)] \\
&= \sum_{i \in T \subset N_1} b_T(C) b_{N_1 - T}(D)[|T| + \beta(y_1)(y_1 - t - 1)] \\
&\qquad + \sum_{i \notin T \subset N_1} b_T(C) b_{N_1 - T}(D)[(|T| + 1) + \beta(y_1 - t)(y_1 - t - 1)] \\
&< \left[\sum_{i \in T \subset N_1} b_T(C) b_{N_1 - T}(D) + \sum_{i \notin T \subset N_1} b_T(C) b_{N_1 - T}(D) \right] (n_1 + \beta(y_1)(y_1 - t - 1)) \\
&= n_1 + \beta(y_1 - t)(y_1 - t - 1) \\
&= F_i(b^*)
\end{aligned}
$$

and for all $j \in N_2$,

$$
\begin{aligned}
&F_j(b) \\
&= \sum_{T \subset N_1} b_T(C) b_{N_1 - T}(D)(|T| + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - p)) \\
&< \left[\sum_{T \subset N_1} b_T(C) b_{N_1 - T}(D) \right] (n_1 + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - p)) \\
&= F_j(b^*).
\end{aligned}
$$

Therefore, the symmetry-invariant equilibrium point b^* payoff-dominates any other symmetry-invariant equilibrium point b. This means that the equilibrium point b^* is a unique solution of $\hat{G}^2(N, N_1, N_2, p)$ in the case of $p = 1 - 1/\beta(y_1 - t)$.

Let us consider the case of $p = 1 - 1/\beta(y_2 + tn_1/n_2)$. In this case, C is the dominant strategy for each high-income individual and the two strategies C and D are indifferent for each low-income individual. Let $d^* = (d_1^*, \ldots, d_n^*)$ be the symmetry-invariant equilibrium point with $d_k = C$ for all $k \in N$ and let $d = (d_1, \ldots, d_n)$ be any other symmetry-invariant equilibrium point. Similarly, we denote by $d_T(a)$ the probability that all individuals in T select the pure strategy a in the equilibrium point d. There exists some $T \subset N_2$ such that $d_T(D) > 0$ since $d^* \neq d$.

Then, we obtain, for all $j \in N_2$,

$$
\begin{aligned}
&E_j(d) \\
&= \sum_{j \in V \subset N_2} d_V(C) d_{N_2 - V}(D)[(n_1 + |V|) + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1)] \\
&\quad + \sum_{j \notin V \subset N_2} d_V(C) d_{N_2 - V}(D)[(n_1 + |V|) + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - p)] \\
&= \sum_{j \in V \subset N_2} d_V(C) d_{N_2 - V}(D)[(n_1 + |V|) + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1)] \\
&\quad + \sum_{j \notin V \subset N_2} d_V(C) d_{N_2 - V}(D)[(n_1 + |V| + 1) + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1)] \\
&< \left[\sum_{j \in V \subset N_2} d_V(C) d_{N_2 - V}(D) + \sum_{j \notin V \subset N_2} d_V(C) d_{N_2 - V}(D) \right] (n + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1)) \\
&= n + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1) \\
&= E_j(d^*),
\end{aligned}
$$

and, for all $i \in N_1$,

$$
\begin{aligned}
&E_i(d) \\
&= \sum_{V \subset N_2} d_V(C) d_{N_2 - V}(D)((n_1 + |V|) + \beta(y_1 - t)(y_1 - t - 1)) \\
&< \left[\sum_{V \subset N_2} d_V(C) d_{N_2 - V}(D) \right] (n + \beta(y_1 - t)(y_1 - t - 1)) \\
&= E_j(d^*).
\end{aligned}
$$

It follows that the symmetric-invariant equilibrium point d^* payoff-dominates the equilibrium point d. □

Proof of Lemma 3: In this stage, there are four possible cases; (a) an agreement on punishment p such that $p \geq 1 - 1/\beta(y_2 + tn_1/n_2)$, (b) an agreement on p such that $1 - 1/\beta(y_1 - t) \leq p < 1 - 1/\beta(y_2 + tn_1/n_2)$, (c) an agreement on p such that $0 \leq p < 1 - 1/\beta(y_1 - t)$ and (d) no agreement. The payoff functions for $i \in N_1$ and $j \in N_2$ in the case of (a) are given by (6) and (7). In the case of (b), the payoff functions are given by (8) and (9). In the case of (c), the payoff functions are given by (10) and (11), and the payoff functions are given by (12) and (13) in the case of (d).

In order to reach a unanimous agreement on the punishment, both high- and low-income individuals must be better off.

Let us consider the case of (a). Since $\beta(\cdot)$ is a decreasing function with y, the income redistribution t may enable us to satisfy the inequality

$$n + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1) \geq \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2).$$

Thus, every low-income individual may benefit from the agreement on the punishment. Moreover, if, for a high-income individual,

$$n + \beta(y_1 - t)(y_1 - t - 1) \geq \beta(y_1 - t)(y_1 - t),$$

then a pair of inequalities (6) $\geq$ (12) and (7) $\geq$ (13) is satisfied, and a unanimous agreement on punishment (a) $p \geq 1 - 1/\beta(y_2 + tn_1/n_2)$ is reached. Thus, if there exists a triplet (t, n_1, n_2) that satisfies the following conditions;

$$n + \beta(y_1 - t)(y_1 - t - 1) > \beta(y_1 - t)(y_1 - t), \text{ and}$$
$$n + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1) > \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2),$$

then a unanimous agreement on punishment $p \geq 1 - 1/\beta(y_2 + tn_1/n_2)$ is reached, and all individuals select cooperation in the action decision stage under this punishment system. As a result, the payoff for a high-income individual in this stage game is given by $n + \beta(y_1 - t)(y_1 - t - 1)$ and that for a low-income individual is given by $n + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1)$.

It is clear that, if a triplet (t, n_1, n_2) does not satisfy the above conditions, no agreement is achieved in the range of punishment (a).

In the case of (b), we obtain from Assumption 1, 2(ii), and 3 that

$$n_1 + \beta(y_1 - t)(y_1 - t - 1) \leq n_1 - \beta(y_1 - t) + \beta(y_1 - t)(y_1 - t) < \beta(y_1 - t)(y_1 - t).$$

Thus, it holds that (8) $<$ (12). Therefore, it is impossible to agree on punishment (b) $1 - 1/\beta(y_1 - t) \leq p < 1 - 1/\beta(y_2 + tn_1/n_2)$. In the case of (c), as long as the punishment level p is strictly positive, we can see that (eq. (10)) $= \beta(y_1 - t)(y_1 - t - p) < \beta(y_1 - t)(y_1 - t) =$ (eq. (12)) and (eq. (11)) $= \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - p) < \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2) =$ (eq. (13)) here. Therefore, an agreement on $0 < p < 1 - 1/\beta(y_1 - t)$ is not reached. If $p = 0$, then it holds that (eq.(10))$=$ (eq.(12)) and (eq.(11))$=$ (eq. (13)). This implies that an agreement on $p = 0$ is possible. If $p = 0$ or no agreement, a unique solution payoff vector $(F_k)_{k \in N}$ in this stage game is given by $F_i = \beta(y_1 - t)(y_1 - t)$ for all $i \in N_1$ and $F_j = \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2)$ for all $j \in N_2$. □

Proof of Proposition 4: Let us consider actions of a low-income individual in this stage game. Every low-income individual has two strategies; 'acceptance (A)' and 'rejection (R)'. If every low-income individual selects A corresponding to some offer t, then each high-income individual proceeds the income redistribution t. On the other hand, if some individuals select R, $t = 0$ is realized, i.e., no income redistribution is proceeded. When $t < 0$ is offered, it can be easily seen that rejection dominates acceptance from the viewpoint of every low-income individual. When $t \geq 0$ is offered, on the contrary, acceptance is a weakly dominant strategy for every low-income individual. Therefore, the key point for income redistribution is whether every high-income individual is better off by proposing $t > 0$.

Let us begin with $t = 0$. Our game is reduced to the state formation game without income redistribution in Section 2.2. In this case, a solution payoff for a high-income individual is y_1.

Next we consider $t > 0$. We can show that every high-income individual has no incentive to offer a redistribution t which does not satisfies the following conditions;

$$n + \beta(y_1 - t)(y_1 - t - 1) > \beta(y_1 - t)(y_1 - t), \text{ and} \tag{17}$$

$$n + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1) > \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2). \tag{18}$$

Even if every high-income individual proposes a t that does not satisfy the above conditions, cooperation does not be ensue in the sequential stage game and the high-income individual has a solution payoff $\beta(y_1 - t)(y_1 - t)$ from Lemma 2 and Lemma 3. Since $\beta(y_1)y_1 > \beta(y_1 - t)(y_1 - t)$, every high-income individual does not offer such an income redistribution. Thus, a high-income individual would offer $t > 0$ only if cooperation is formed, i.e., $t \in [0, (y_1 - y_2)n_2/n]$ satisfies inequalities (17) and (18), and he/she obtain his/her payoff more than y_1. Therefore, if there exists $t \in [0, (y_1 - y_2)n_2/n]$ that satisfies the following conditions

$$n + \beta(y_1 - t)(y_1 - t - 1) \geq \beta(y_1)y_1, \text{ and}$$

$$n + \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2 - 1) \geq \beta(y_2 + tn_1/n_2)(y_2 + tn_1/n_2),$$

every high-income individual offers the minimum t^* which satisfies the above conditions and every low-income individual accepts their offers. Since cooperation is formed in the sequential stage game under t^*, this game results in the solution payoff vector $(F_k)_{k \in N}$ such that $F_i = n + \beta(y_1 - t^*)(y_1 - t^* - 1)$ for all $i \in N_1$ and $F_j = n + \beta(y_2 + t^* n_1/n_2)(y_2 + t^* n_1/n_2 - 1)$ for $j \in N_2$. This completes the proof. □

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