Existence of a Stationary Subgame Perfect Equilibrium in a Coalitional Bargaining Model with Non-superadditive Game

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Abstract

We prove the existence of a stationary subgame perfect equilibrium (SSPE) in a noncooperative coalitional bargaining game model with random proposers. Our model contains the bargaining situation in which the coalitional form game is not superadditive. We also provide a necessary and sufficient condition to exist a pure-strategy SSPE satisfying the efficiency property as a discount factor is close to one.

JEL Classification Numbers: C72, C78.

1 Introduction

This paper presents an existence proof of a stationary subgame perfect equilibrium (SSPE) in an n-person noncooperative coalitional bargaining game with random proposers. A noncooperative coalitional bargaining model with random proposers has been provided by Okada (1996). He considered a bargaining situation described by an n-person coalitional game (N, v) with transferable utility, where the characteristic function v of the game is superadditive. We extend the situation to that not necessarily described by a superadditive game (N, v). The non-superadditive game (N, v) includes a local public goods economy with congestion effects, which will be taken for example later. Mutuswami, Perez-Castrillo and Wettstein (2004) have

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also shown that the game associating to a local public goods economy is not necessarily superadditive.

A noncooperative coalitional bargaining model is represented as an infinite-length extensive game with perfect information. It is well-known that an infinite-length extensive form game does not have a Nash equilibrium at all. Therefore, we have to prove the existence of an SSPE in a noncooperative coalitional bargaining model with random proposers. In Okada (1996)'s paper, the existence proof itself was not given in a general setting, but a necessary and sufficient condition for a pure strategy SSPE with the efficiency property is provided. We also clarify a necessary and sufficient condition for a pure-strategy efficient SSPE to exist in the random-proposers bargaining model for a coalitional game which contains the non-superadditive case. The necessary and sufficient condition is related to the notion of a *C-stable solution* or a *Core of coalition structure*, which is introduced by Guesnerie and Oddou (1979, 1981) and Greenberg and Weber (1986).

The paper is organized as follows. In Section 2, we present a noncooperative coalitional bargaining game model with random proposers, and give an example of the non-superadditive coalitional form game. In Section 3, we provide two existence theorems for a stationary subgame perfect equilibrium point of the bargaining game. Proofs of theorems in Section 3 are gathered in Section 4.

2 Random-proposers Model

The bargaining situation is described by an n-person coalitional form game (N, v) with transferable utility. Here, $N = \{1, \ldots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is the characteristic funtion. The characteristic function v is assumed to be 0-*normalized* ($v(\{i\}) = 0$ for all $i \in N$) and *essential* ($v(N) > 0$). We allow the characteristic function to be *non-superadditive*; for some $S, T \subset N$ such that $S \cap T = \emptyset$, $v(S \cup T) < v(S) + v(T)$. It is called a coalition structure $\{N\}$ to be *universally efficient* if for every partition $\{S_1, \ldots, S_K\}$ of N, $v(N) \geq \sum_{k=1}^{K} v(S_k)$. If the game is superadditive, the grand coalition N is universally efficient. But the grand coalition N is not necessary universally efficient in our situation.

Example. Let consider a two-goods economy consisting of one local public good and one private good. We assume that the production technology permits the transformation of one unit of private good into one unit of public good. Each individual $i \in N$ is endowed with a same amount I of private good and has a quasi-linear utility function with congestion effects:

$$u_i(g) - c(|S|) + x_i,$$

where $u_i(g)$ is utility from the local public good g and x_i is the consumption of private good. The term $c(|S|)$ is disutility from congestion, where $|S|$ denoted the cardinality of the coalition S. The coalitional form game (N, v) associating to the local public good economy is defined naturally such as a market game by; for each $S \subset N$,

$$v(S) = \max_{g \in \mathbb{R}} \left\{ \sum_{i \in S} u_i(g) - |S|c(|S|) + |S|I - g \right\}.$$

The value $v(S)$ denotes the maximum total payoff of the members of coalition S by producing the local public good. Since each individual has a quasi-linear utility function, the game (N, v) is a game in coalitional form with transferable utility.

Let us give an example to show the local public good game (N, v) to be non-superadditive. $N = \{1, 2, 3, 4\}$. $I = 5$. Each utility function of the public good is given by: $u_1(g) = u_2(g) = 4 \log g$ and $u_3(g) = u_4(g) = 2 \log g$. Congestion costs are represented by: $c(1) = 1$, $c(2) = 3$, $c(3) = 10$, and $c(4) = 15$. Then, the value of characteristic function for each coalition is:

$$\begin{aligned}
&v(\{1\}) = v(\{2\}) = 8 \log 2 = 5.5548 \cdots, \\
&v(\{3\}) = v(\{4\}) = 2 \log 2 + 2 = 3.33862 \cdots, \\
&v(\{1,2\}) = 24 \log 2 - 1 = 15.6344 \cdots, \\
&v(\{3,4\}) = 8 \log 2 + 3 = 8.5448 \cdots, \\
&v(\{1,2,3\}) = v(\{1,2,4\}) = 10 \log 10 - 5 = 18.025 \cdots, \\
&v(\{1,3,4\}) = v(\{2,3,4\}) = 24 \log 2 - 3 = 13.6344 \cdots, \\
&v(\{1,2,3,4\}) = 12 \log 12 - 7 = 22.8176 \cdots.
\end{aligned}$$

Therefore, we can get

$$v(\{1,2\}) + v(\{3,4\}) > v(\{1,2,3,4\}).$$

Thus, v is not superadditive, and the grand coalition N is not universally efficient in this example.

Next, let explain the *random-proposer* model of bargaining. A payoff vector for a coalition S is denoted by $y^S = (y_i^S)_{i \in S} \in \mathbb{R}^{|S|}$. A payoff vector y^S for S is called *feasible* if

$$\sum_{i \in S} y_i^S \leq v(S).$$

We denote by Y^S the set of all feasible payoff vectors for S.

Our noncooperative bargaining model proceeds as follows. At every round $t = 1, 2, \ldots$, one player is selected as a proposer with equal probability among all player still active in bargaining. Let N^t be the set of all active players at round t, where the bargaining starts with all players at round 1, i.e., $N^1 = N$. The proposer i chooses a coalition S with $i \in S \subset N^t$ and a payoff vector $y^S \in Y^S$. All other players in S accept or reject the proposal sequentially. If all the other players in the coalition accept the proposal, then it is agreed upon and enforced. The remaining players outside S can continue negotiations at the next round. Thus, $N^{t+1} = N^t \backslash S$. If some players in S reject the proposal, then negotiations go on to the next round and a new proposer randomly selected by the same rule. In this case, $N^{t+1} = N^t$. The bargaining continues until there exists a subset S of active players such that $v(S) > 0$.

When a proposal (S, y^S) is agreed upon at round t, the payoff of every member $i \in S$ is $\delta^{t-1} y_i^S$, where δ is a discount factor, and $0 \leq \delta < 1$. For players who do not belong to any coalitions, their payoffs are assumed to be zero. Every player has perfect information.

Our model is formally represented as an infinite-length extensive form game with perfect information and with chance moves. We denote by $\Gamma^S(\delta)$ the bargaining model with the player set $S \subset N$. Γ^S is used when the discount factors δ converge to one. Let σ_i be a strategy for player i in $\Gamma^N(\delta)$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a strategy combination. The solution concept that we shall apply to our bargaining model is a *stationary subgame perfect equilibrium point* (SSPE).

Definition 1. (i) A strategy combination σ^* of the game $\Gamma^N(\delta)$ is called a *stationary subgame perfect equilibrium point* (SSPE) if it is a subgame perfect equilibrium point with the property that for every $t = 1, 2, \ldots$, the tth round strategy of every player depends only on the set N^t of all players active at round t ant the proposal at round t. (ii) A strategy combination σ^* of the game Γ^N is called a *limit* SSPE if it is a limit point of SSPEs of $\Gamma^N(\delta)$ as $\delta \to 1$[1].

For an SSPE σ of $\Gamma^N(\delta)$ and every coalition $S \subset N$, let $v^S = (v_i^S)_{i \in S}$ denotes the expected payoff vector of players for σ in the subgame $\Gamma^S(\delta)$, and $\theta^S = (T_i^S)_{i \in S}$ be the collection of coalitions T_i^S proposed by every player

[1]Precisely speaking, a limit SSPE is defined by the limit point of the equilibrium payoff configuration $(v^S(\delta))_{S \subset N}$ of $\Gamma^N(\delta)$ as $\delta \to 1$. Thus, $v^S(\delta) \to v^S$ as $\delta \to 1$, where $v^S(\delta) \in \mathbb{R}^S$. We can easily construct a SSPE strategy combination which supports the limit point $(v^S)_{S \subset N}$ of the payoff configuration.

i on the plays of σ in $\Gamma^S(\delta)$. We call the collection $\{(v^S, \theta^S) \mid S \subset N\}$ the *configuration* of the SSPE σ.

3 Existence Theorems

Let us establish the existence theorem for an SSPE in the general random-proposer bargaining model. If mixed strategies about the proposal of a coalition are allowed, the existence of an SSPE is guaranteed. All proofs of theorems are gathered in Section 4.

Theorem 1. *If mixed strategies about the choice of a coalition by each proposer are allowed, there exists a stationary subgame perfect equilibrium point of the game $\Gamma^N(\delta)$.*

Denote the set of all partitions of S by

$$\Pi(S) = \left\{ \{S_1, \ldots, S_K\} \;\middle|\; \bigcup_{k=1}^{K} S_k = S, \text{ and } S_i \cap S_j = \emptyset, i \neq j \right\}.$$

A element $\pi^S \in \Pi(S)$ is called a *coalition structure* of S. For each coalition structure $\pi^S = \{S_1, \ldots, S_K\}$ of S, the function on $\Pi(S)$ is defined by

$$V(\pi^S; S) = \sum_{k=1}^{K} v(S_k).$$

Definition 2. A coalition structure π is called an *efficient coalition structure* of S if $V(\pi; S) \geq V(\pi'; S)$ for all $\pi' \in \Pi(S)$.

Let define the efficiency of a SSPE for Γ^N.

Definition 3. An SSPE σ of the game $\Gamma^N(\delta)$ is called *subgame coalitional efficient* if, for every subgame $\Gamma^S(\delta)$, every player $i \in S$ proposes the coalition which is a component of the efficient coalition structure of S in σ. A *limit subgame coalitional efficient* SSPE of Γ^N is defined to be a limit point of subgame coalitional efficient SSPEs of $\Gamma^N(\delta)$ as $\delta \to 1$.

The notion of subgame coalitional efficiency requires that in all subgames $\Gamma^S(\delta)$ the efficient coalition structure is formed. This notion is stronger than the Pareto efficiency of the expected payoff vector for n players in $\Gamma^N(\delta)$.

The next theorem (Theorem 2) characterizes the situation in which there exists a *pure strategy* and limit subgame coalitional efficient SSPE in Γ^N. Note that the existence of a pure-strategy SSPE is investigated. In Theorem 2, the efficient coalition structure of each S is expressed by $\pi^*(S) = \{S_1^*(S), \ldots, S_{K^S}^*(S)\}$.

Theorem 2. *There exists a pure strategy and limit subgame coalitional efficient SSPE of* Γ^N *if and only if the game* (N, v) *satisfies; for all* $S \subset N$,

$$\frac{v(S_1^*(S))}{|S_1^*(S)|} \geq \max_{T \subset N, i \in T} \left(v(T) - \sum_{j \in T, j \neq i} y_j \right) \text{sub.to } y_j = \frac{v(S_k^*(S))}{|S_k^*(S)|}, \tag{1}$$
$$j \in S_k^*(S) \cap T, k = 1, 2, \dots, K^S, \ \textit{for all } i \in S_1^*(S),$$
$$\vdots$$
$$\frac{v(S_{K^S}^*(S))}{|S_{K^S}^*(S)|} \geq \max_{T \subset N, i \in T} \left(v(T) - \sum_{j \in T, j \neq i} y_j \right) \text{sub.to } y_j = \frac{v(S_k^*(S))}{|S_k^*(S)|},$$
$$j \in S_k^*(S) \cap T, k = 1, 2, \dots, K^S, \ \textit{for all } i \in S_{K^S}^*(S).$$

The expected equilibrium payoff vector $(v_j^*)_{j \in S}$ *in* Γ^S *is given by*

$$v_i^* = \frac{v(S_1^*(S))}{|S_1^*(S)|}, \ \forall \ i \in S_1^*(S), \dots, v_i^* = \frac{v(S_{K^S}^*(S))}{|S_{K^S}^*(S)|}, \ \forall \ i \in S_{K^S}^*(S). \tag{2}$$

Theorem 2 shows that a pure strategy and limit subgame coalitional efficient SSPE exists if and only if each individual obtains the maximum payoff by forming coalition $S_k^*(S)$ under the condition that other individual j must be guaranteed to get his payoff $v(S_\ell^*(S))/|S_\ell^*(S)|$, where $S_\ell^*(S) \in \pi^*(S)$ and $j \in S_\ell^*(S)$, if j is the member of coalition.

We introduce the notion of a *C-stable solution*, which is considered in Guesnerie and Oddou (1979, 1981) and Greenberg and Weber (1986).

Definition 4. A *C-stable solution* of (S, v) is a payoff vector $y \in \mathbb{R}^{|S|}$, which satisfies the following properties:

> there exists a coalition structure $\pi = \{S_1, \dots, S_K\} \in \Pi(S)$ such that $\sum_{j \in S_k} y_j \leq v(S_k)$ for all $k = 1, 2, \dots, K$ and that y does not blocked by a coalition T; there is no $T \subset S$ such that $\sum_{j \in T} y_j' \leq v(T)$ and $y_j' \geq y_j$ for all $j \in T$.

The notion of a C-stable solution is an extension of the core concept. If a coalition structure π is assumed to be $\{N\}$, then the C-stable solution implies the Core.

There is a relationship between Theorem 2 and a C-stable solution. Fix a coalition S. The efficient coalition structure of S is given by $\pi^*(S) = \{S_1^*(S), \dots, S_{K^S}^*(S)\}$. For each $S_\ell^*(S)$, $\ell = 1, \dots, K^S$, the sum of money $v(S_\ell^*(S))$ distributes among the members of the coalition $S_\ell^*(S)$ equally. Thus, player $j \in S_\ell^*(S)$ receives the payoff of $v(S_\ell^*(S))/|S_\ell^*(S)|$. The above payoff allocation for the member of coalition $S^*_\ell(S)$ is coincident with a Nash bargaining solution of the bargaining problem $(S^*_\ell(S), v(S^*_\ell(S)))$. The Nash bargaining solution of the problem $(S^*_\ell(S), v(S^*_\ell(S))$ is defined as the solution of the maximization problem:

$$\max_{(y_i)_{i\in S^*_\ell(S)}} \prod_{j\in S^*_\ell(S)} y_j \text{ subject to } \sum_{j\in S^*_\ell(S)} y_j \leq v(S^*_\ell(S)).$$

We have to note that the condition (1) in Theorem 2 says that, for every coalition S of N, the above payoff allocation is in the C-stable solution of (S, v).

4 Proofs

4.1 Proof of Theorem 1

This existence theorem is proved in the same line as the proof of Theorem 2.1 in Ray and Vohra (1999).

The proof is given by induction on the number of players. The theorem holds trivially for the 1-player case. We assume that it holds for the less than n-players case. This hypothesis implies that there exists an SSPE for every subgame $\Gamma^{N\setminus S}(\delta)$ for every nonempty coalition S. We fix one equilibrium strategy combination for each subgame. Let $v^{N\setminus S} = (v_i^{N\setminus S})_{i\in N\setminus S}$ be the expected payoff vector in $\Gamma^{N\setminus S}(\delta)$. What we have to do is to describe equilibrium strategies for all the remaining nodes in the game $\Gamma^N(\delta)$.

Let us introduce several notations. Let $\mathcal{S}_i = \{S \subset N \mid i \in S\}$ be the set of all nonempty coalitions containing player i, and let Δ_i be the set of probability distributions over the choice set $A_i = (\mathcal{S}_i, (\{j\})_{j\in N\setminus\{i\}})$. We define $\Delta = \prod_{i\in N} \Delta_i$. The choice set A_i indicates that player i make an acceptable proposals to a coalition in $\mathcal{S}_i$ or an unacceptable proposal to player j. Now α_i denote player i's choice of coalitions to form or other players to whom an unacceptable offer is made. $\alpha_i(S)$ represents the probability with which i chooses to make an acceptable proposal to S, and $\alpha_i(\{j\})$ is the probability with which i choose to make an unacceptable proposal to player j. We assumed that $v(\{i\}) \geq 0$ for all $i \in N$ and a payoff to every player is bounded above by $m = \max_{S\subset N} v(S)$. Therefore, we restrict the feasible payoff vectors to $X = [0, m]^n$ in searching for equilibrium payoffs.

Fix a vector $\alpha \in \Delta$ and a vector $x^i = (x^i_1, \ldots, x^i_n) \in X$. Define $v_i^N = \sum_{k\in N} x_i^k/n$ for every i. Because the set X is convex, the convex combination $v^N = (v_i^N)_{i\in N}$ is in X. As we see below, the vector x^i is interpreted as the vector of expected equilibrium payoffs that each player receives if player i

becomes the proposer at round 1, and, by the definition of $\Gamma^N(\delta)$, the vector v^N becomes the expected equilibrium payoff vector.

First, i can propose a coalition $S \in \mathcal{S}_i$ and a payoff vector $y^i(S)$ if he becomes the proposer at round 1. Consider the following problem:

$$\max_y \; y_i \; \text{ subject to } y_j \geq \delta v_j^N, j \in S, j \neq i, \text{ and, } \sum_{j \in S} y_j \leq v(S). \qquad (3)$$

Let $g_i(S, v^N)$ be the maximum value which is attained in the above problem. It satisfies

$$g_i(S, v^N) = v(S) - \sum_{j \in S, j \neq i} \delta v_j^N.$$

It is easy to see that $g_i(S, v^N)$ is a continuous function of v^N.

Second, i might make an unacceptable proposal to j. (In this case, i obtains the expected payoff δv_i^N.)

Let compute a present value payoff to player i under a given $(v^N, \alpha) \in X \times \Delta$. We first define a function on $X \times \Delta$ by

$$v_i^j(v^N, \alpha) = B_i^j + \delta \sum_{\ell \in N, \ell \neq j} \alpha_j(\{\ell\}) \sum_{k \in N} \frac{v_i^k(v^N, \alpha)}{n}, \qquad (4)$$

for all $j \in N$, where

$$B_i^i = \sum_{S \in \mathcal{S}_i} \alpha_i(S) g_i(S, v^N), \text{ and,}$$

$$B_i^j = \left[\sum_{S \in \mathcal{S}_j, i \in S} \alpha_j(S) \right] \delta v_i^N + \sum_{S \in \mathcal{S}_j, i \notin S} \alpha_j(S) \delta v_i^{N \setminus S}.$$

The equations system (4) can be written in matrix form as $EV_i^T = B_i^T$, where $V_i = [v_i^1(v^N, \alpha), v_i^2(v^N, \alpha), \ldots, v_i^n(v^N, \alpha)]$, $B_i = [B_i^1, B_i^2, \ldots, B_i^n]$, and

$$E = \begin{bmatrix} 1 - \delta \sum_{\ell \neq 1} \frac{\alpha_1(\{\ell\})}{n} & -\delta \sum_{\ell \neq 1} \frac{\alpha_1(\{\ell\})}{n} & \ldots & -\delta \sum_{\ell \neq 1} \frac{\alpha_1(\{\ell\})}{n} \\ -\delta \sum_{\ell \neq 2} \frac{\alpha_2(\{\ell\})}{n} & 1 - \delta \sum_{\ell \neq 2} \frac{\alpha_2(\{\ell\})}{n} & \ldots & -\delta \sum_{\ell \neq 2} \frac{\alpha_2(\{\ell\})}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\delta \sum_{\ell \neq n} \frac{\alpha_n(\{\ell\})}{n} & -\delta \sum_{\ell \neq n} \frac{\alpha_n(\{\ell\})}{n} & \ldots & 1 - \delta \sum_{\ell \neq n} \frac{\alpha_n(\{\ell\})}{n} \end{bmatrix}.$$

Note that E is the nonsingular and dominant diagonal matrix. Therefore, $v_i^j(v^N, \alpha)$ is continuous in v^N and α for all $j \in N$.

Next define a function on $X \times \Delta \times \Delta_i$ by

$$\xi_i(v^N, \alpha, \tilde{\alpha}_i) = \sum_{S \in \mathcal{S}_i} \tilde{\alpha}_i(S) g_i(S, v^N) + \delta \sum_{\ell \neq i} \tilde{\alpha}_i(\{\ell\}) \sum_{k \in N} \frac{v_i^k(v^N, \alpha)}{n}.$$

and let denote

$$\psi_i^1(v^N, \alpha) = \arg\max_{\tilde{\alpha}_i \in \Delta_i} \xi_i(v^N, \alpha, \tilde{\alpha}_i),$$
$$\psi_i^2(v^N, \alpha) = \xi_i(v^N, \alpha, \psi_i(v^N, \alpha)).$$

By Berge's maximum theorem, the correspondence $\psi_i^1(v^N, \alpha)$ is upper hemicontinumous and convex-valued ant the function $\psi_i^2(v^N, \alpha)$ is continuous. It follows from $v(\{i\}) > 0$ for all i that $\psi_i^2(v^N, \alpha) \in [0, m]$ for all i. In addition, the set $X \times \Delta$ is compact and convex. Therefore, the correspondence

$$\psi = \prod_{i=1}^{n} \psi_i^2 \times \prod_{i=1}^{n} \psi_1^1 : X \times \Delta \to X \times \Delta.$$

is upper hemicontinuous, nonempty and convex-valued on the compact and convex domain. By Kakutani's fixed point theorem, we have a fixed point (v^{N*}, α^*) such that $(v^{N*}, \alpha^*) \in \psi(v^{N*}, \alpha^*)$.

Using the fixed point (v^{N*}, α^*), we can construct an SSPE of the game $\Gamma^N(\delta)$. Define the strategy combination $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that: (a) When the set of active players is N, player i proposes coalitions according to α_i^* and a payoff vector $y(S)$ which solves the problem defined by (4.1) to every coalition $S \in \mathcal{S}_i$ such that $\alpha_i^*(S) > 0$. To every j such that $\alpha_i^*(\{j\}) > 0$, she offers less than δv_j^{N*}. (b) When the set of active players is N, player i as a responder accepts the proposal $y_i(S)$ if and only if $y_i(S) \geq \delta v_i^{N*}$. (c) When the set of active players is not N, the strategies of the active players are defined by a pre-selected equilibrium of the game $\Gamma^{N \setminus S}(\delta)$.

We can check that the strategy combination σ satisfying (a), (b) and (c) is an SSPE of $\Gamma^N(\delta)$. By construction, it is satisfied that $v_i^{N*} = \xi_i(v^{N*}, \alpha^*, \alpha_i^*) = \max \xi_i(v^{N*}, \alpha^*, \cdot)$. This implies that player i as a proposer could not receive a higher payoff than v_i^N by deviating from α_i^*. The action prescribed in (a) achieves v_i^{N*}. Thus, it is the optimal strategy as a proposer for player i. Because i's continuation payoff is δv_i^{N*}, the strategy in (b) as a responder is trivially optimal. Finally, if some players left the game, the actions in (c) are optimal strategies by induction hypothesis. Thus, σ is a SSPE of $\Gamma^N(\delta)$.

4.2 Proof of Theorem 2

We provide two lemmas before giving the proof of Theorem 2. In addition, we focus on a class of payoff configuration in these lemmas.

Definition 5. A payoff configuration $\{v^S \mid S \subset N\}$ is called *feasible in the efficient coalition structure* $\pi^*(S) = \{S_1^*(S), \ldots, S_{K^S}^*\}$ if, for every S,

$$\sum_{j \in S_\ell^*(S)} v_j^S \leq v(S_\ell^*(S)), \quad \ell = 1, \ldots, K^S.$$

The first lemma shows that, in every pure strategy SSPE those payoff configuration is feasible in the efficient coalition structure, an agreement is made in the first round. We have to restrict a class of SSPE to prove no delay of agreement in equilibrium. If a game is superadditive, this restriction is unnecessary; no delay occurs in equilibrium, see Okada (1996).

Lemma 1. *In every pure strategy SSPE* σ *of* $\Gamma^N(\delta)$ *with* $\{(v^S, \theta^S) \mid S \subset N\}$ *such that the payoff configuration is feasible in the efficient coalition structure, every player* $i \in N$ *proposes at round 1 a solution* (S_i, y^{S_i}) *of the maximization problem:*

$$\max_{S, y^i} \left(v(S) - \sum_{j \in S, j \neq i} y_j^i \right) \tag{5}$$
$$\text{subject to } y_j^i \geq \delta v_j^N, \text{ for all } j \in S, j \neq i,$$
$$S \in \mathcal{S}_i.$$

Moreover, the proposal (S_i, y^{S_i}) *is accepted in* σ.

Proof. Let $x^i = (x_1^i, \ldots, x_n^i)$ be the expected equilibrium payoff vector when player i becomes the proposer at round 1. By definition of $\Gamma^N(\delta)$, $v_i^N = \sum_{k \in N} x_i^k / n$ for all $i \in N$. We denote by m^i the maximum value of (5).

We will prove that $x_i^i = m^i$.

($x_i^i \leq m^i$): Suppose that player i proposes (S, y^S) at round 1 such that $y_i^S > m^i$. Since m^i is the maximum value of (5), for some $j \in S$ with $j \neq i$, $y_j^S < \delta v_j^N$. Let j^* be the last responder of such a kind. In equilibrium the following is possible: (i) some responder after j^* reject i's proposal, and (ii) otherwise. If player j accepts the proposal in the case of (ii), then the proposal is agreed upon and player j obtain $y_{j^*}^S$ less than his continuation payoff δv_j^N. Therefore, it is optimal for j^* to reject i's proposal. Thus, i's proposal is rejected and the game goes on to round 2 whichever case occurs. Then, player i obtains the discount payoff δv_i^N.

The efficient coalition structure of N is denoted by $\pi^*(N) = \{S_1^*(N), \ldots, S_{K^N}^*\}$. Because we focus on a SSPE those payoff configuration is feasible in the efficient coalition structure, we have

$$\sum_{j \in S_\ell^*(N)} v_j^N \leq v(S_\ell^*(N)), \text{ for } \ell = 1, \ldots, K^N.$$

Thus, the pair $(S_\ell^*(N), (v_j^N)_{j\in S_\ell^*(N)})$, where $i \in S_\ell^*(N)$, satisfies constraints of the problem (5). This implies that $v_i^N \le m^i$.

Since $V(\{i\}) \ge 0$ for all $i \in N$, every player i surely obtain more than zero as a payoff when i becomes the proposer. Therefore, $x_i^i \ge 0$. The responder also rejects the proposal in which his payoff is less than zero. Thus, $x_k^i \ge 0$. We have $v_i^N \ge 0$ because the above argument applies to all $i \in N$. Hence, $\delta v_i^N \le v_i^N \le m^i$. Player i obtains only δv_i^N even if he proposes a payoff greater than m^i. This implies $x_i^i \le m^i$.

($x_i^i \ge m^i$): By the assumption that the payoff configuration is feasible in the efficient coalition structure, the pair $(S_\ell^*(N), (v_j^N)_{j\in S_\ell^*(N)})$ is a feasible solution of the problem (5). Then, the pair $(S_\ell^*(N), (\delta v_j^N)_{j\in S_\ell^*(N)})$ is also a feasible solution of the problem (5) because $0 \le \delta \le 1$. Therefore, $m^i \ge v_i^N \ge \delta v_i^N$. Suppose that $m^i = 0$. Then, $v_i^N = 0$, and the payoff combination $(0, (\delta v_j^N)_{j\in S_\ell^*(N)\setminus\{i\}})$ is feasible for S; $\sum_{j\in S_\ell^*(N)\setminus\{i\}} \delta v_j^N \le v(S_\ell^*(N))$. Two cases are possible: (i) If $v_j^N = 0$ for all $j \in S_\ell^*(N)\setminus\{i\}$, then there exists a feasible payoff combination $(y_i, (\delta v_j^N)_{j\in S_\ell^*(N)\setminus\{i\}})$ such that $y_i > 0$ because $v(S_\ell^*(N)) > 0$. (ii) If $v_j^N > 0$ for some $j \in S_\ell^*(N)\setminus\{i\}$, then $\delta v_j^N < v_j^N$. Thus, some $(y_j, (\delta v_j^N)_{j\in S_\ell^*(N)\setminus\{i\}})$, where $y_j > 0$, become feasible solution of the problem (5). Because m^i is the maximum value of (5), we must have $m^i > 0$. Any solution (S, y^S) of (5) satisfies $y_i^S = m^i$ and $y_j^S = \delta v_j^N$ for $j \in S$, $j \ne i$. For any $\varepsilon > 0$, define z^S such that

$$z_i^S = m^i - \varepsilon, \quad z_j^S = y_j^S + \frac{\varepsilon}{|S|-1}, j \in S, j \ne i.$$

If player i proposes (S, z^S), then it is accepted by all $j \in S$, $j \ne i$. Therefore, $x_i^i \ge z_i^S = m^i - \varepsilon$. Since ε is arbitrary, we conclude $x_i^i \ge m^i$.

Finally, we show that i's proposal is accepted at round 1. It is sufficient to prove $\delta v_i^N < m^i$. Suppose that $\delta v_i^N = m^i$. It follows from $\delta v_i^N \le v_i^N \le m^i$ that $m^i = v_i^N = 0$. This contradicts with $m^i > 0$. □

We next present a necessary and sufficient condition for the existence of a pure strategy SSPE of $\Gamma^N(\delta)$. The corresponding theorem in the case of a superadditive game is given in Okada (1996).

Lemma 2. *For $\psi = \{(v^S, \theta^S) \mid S \subset N\}$, where $v^S = (v_i^S)_{i\in S}$, $\theta^S = (T_i^S)_{i\in S}$ and the payoff configuration $\{v^S \mid S \subset N\}$ is feasible in the efficient coalition structure, there exists a pure strategy SSPE σ of $\Gamma^N(\delta)$ with ψ if and only if, for every $S \subset N$ and for every $i \in S$,* *(i) the coalition T_i^S constitutes a solution of*

$$\max_{T,y^i} \left(v(T) - \sum_{j \in T, j \neq i} y_j^i \right) \tag{6}$$

subject to $y_j^i \geq \delta v_j^N$, *for all* $j \in S, j \neq i$, *and* $T \in \mathcal{S}_i$.

(ii) the expected payoff vector $v^S = (v_i^S)_{i \in S}$ *satisfies*

$$v_i^S = \frac{1}{|S|} \left\{ v(T_i^S) - \delta \sum_{j \in T_i^S, j \neq i} v_j^S \right\} + \frac{1}{|S|} \sum_{k: i \in T_k^S, k \neq i} v_i^S + \frac{1}{|S|} \delta \sum_{m: i \notin T_m^S} v_i^{S \setminus T_m^S}, \tag{7}$$

where v_i^T *is defined to be zero when* $T = \emptyset$.

Proof. (only if): Let σ be a SSPE of $\Gamma^N(\delta)$ with $\psi = \{(v^S, \theta^S) \mid S \subset N\}$. We can apply Lemma 1 to every subgame $\Gamma^S(\delta)$. Then, (i) is proved. In the subgame $\Gamma^S(\delta)$, every player i makes a proposal of the payoff allocation $x^i = (x_j^i)_{j \in T_i^S}$ such that

$$x_i^i = v(T_i^S) - \sum_{j \in T_i^S, j \neq i} \delta v_j^S, \quad x_j^i = \delta v_j^S, \ j \in T_i^S, j \neq i. \tag{8}$$

Since this proposal is accepted at round 1, we can obtain (7) by the definition of $\Gamma^S(\delta)$.

(if): Define the strategy combination $\sigma = (\sigma_1, \ldots, \sigma_n)$ of $\Gamma^N(\delta)$ such that, in every subgame $\Gamma^S(\delta)$, every player $i \in S$ proposes a solution (T_i^S, x_i) of the problem (6) satisfying (8), and accepts any proposal (T, y^T) if and only if $y_i^T \geq \delta v_i^S$. It is easy to see that σ is a SSPE of $\Gamma^N(\delta)$ with ψ. □

Let us now turn to the proof of Theorem 2. By the definition of subgame coalitional efficient SSPE, the payoff configuration of the equilibrium is feasible in the efficient coalition structure. Therefore, we can use Lemma 1 and 2 to prove Theorem 2.

Proof. (only if): Let denote the efficient coalition structure of S by $\pi^*(S) = \{S_1^*(S), \ldots, S_{K^S}^*(S)\}$. Assume that there exists a pure strategy and limit subgame coalitional efficient SSPE of Γ^N. It follows from (ii) of Lemma 2 that, for every coalition $S \subset N$ and for every $i \in S^*_\ell(S)$, $\ell = 1, \ldots, K^S$,

$$v_i^S = \frac{1}{|S|}\left\{ v(S_1^*(S)) - \delta \sum_{k \in S_1^*(S), k \neq i} v_k^S \right\} + \frac{|S_1^*(S)| - 1}{|S|} \delta v_i^S$$
$$+ \frac{|S_2^*(S)|}{|S|} \delta v_i^{S \setminus S_2^*(S)} + \cdots + \frac{|S_{K^S}^*(S)|}{|S|} \delta v_i^{S \setminus S_{K^S}^*(S)}, \text{ for all } i \in S_1^*(S),$$
$$\vdots$$
$$v_i^S = \frac{1}{|S|}\left\{ v(S_{K^S}^*(S)) - \delta \sum_{k \in S_{K^S}^*(S), k \neq i} v_k^S \right\} + \frac{|S_{K^S}^*(S)| - 1}{|S|} \delta v_i^S$$
$$+ \frac{|S_1^*(S)|}{|S|} \delta v_i^{S \setminus S_1^*(S)} + \cdots + \frac{|S_{K^S-1}^*(S)|}{|S|} \delta v_i^{S \setminus S_{K^S-1}^*(S)}, \text{ for all } i \in S_{K^S}^*(S).$$

The above equations system is uniquely solvable for any $\delta < 1$, and the solution $(v_i^S)_{i \in S}$ is expressed in the following recursive form:

(i) For each element of $\pi^*(S)$,

$$v_i^{S_1^*(S)} = \frac{v(S_1^*(S))}{|S_1^*(S)|}, \text{ for all } i \in S_1^*(S),$$
$$\ldots, v_i^{S_{K^S}^*(S)} = \frac{v(S_{K^S}^*(S))}{|S_{K^S}^*(S)|}, \text{ for all } i \in S_{K^S}^*(S),$$

(ii) For each union of two elements of $\pi^*(S)$,

$$v_i^{S_1^*(S) \cup S_2^*(S)} = \frac{1}{|S_1^*(S) \cup S_2^*(S)|}\left[v(S_1^*(S)) + |S_2^*(S)| \delta v_i^{S_1^*(S)} \right], \text{ for all } i \in S_1^*(S),$$
$$v_i^{S_1^*(S) \cup S_2^*(S)} = \frac{1}{|S_1^*(S) \cup S_2^*(S)|}\left[v(S_2^*(S)) + |S_1^*(S)| \delta v_i^{S_2^*(S)} \right], \text{ for all } i \in S_2^*(S),$$
$$\vdots$$
$$v_i^{S_{K^S-1}^*(S) \cup S_{K^S}^*(S)} = \frac{1}{|S_{K^S-1}^*(S) \cup S_{K^S}^*(S)|}\left[v(S_{K^S-1}^*(S)) + |S_{K^S}^*(S)| \delta v_i^{S_{K^S-1}^*(S)} \right],$$
$$\text{for all } i \in S_{K^S-1}^*(S),$$
$$v_i^{S_{K^S-1}^*(S) \cup S_{K^S}^*(S)} = \frac{1}{|S_{K^S-1}^*(S) \cup S_{K^S}^*(S)|}\left[v(S_{K^S}^*(S)) + |S_{K^S-1}^*(S)| \delta v_i^{S_{K^S}^*(S)} \right],$$
$$\text{for all } i \in S_{K^S}^*(S),$$

$$\vdots$$

(iii) As a result, we have, for $S = S_1^*(S) \cup \cdots \cup S_{K^S}^*(S)$,

$$v_i^S = \frac{1}{|S|}\left[v(S_1^*(S)) + |S_2^*(S)|\delta v_i^{S\setminus S_2^*(S)} + \cdots + |S_{K^S}^*(S)|\delta v_i^{S\setminus S_{K^S}^*(S)}\right], \quad \text{for all } i \in S_1^*(S),$$

$$\vdots$$

$$v_i^S = \frac{1}{|S|}\left[v(S_{K^S}^*(S)) + |S_1^*(S)|\delta v_i^{S\setminus S_1^*(S)} + \cdots + |S_{K^S-1}^*(S)|\delta v_i^{S\setminus S_{K^S-1}^*(S)}\right], \quad \text{for all } i \in S_{K^S}^*(S).$$

It is easy to see that, for every $S \subset N$ and for the efficient coalition structure $\pi^*(S) = \{S_1^*(S), \ldots, S_{K^S}^*(S)\}$,

$$v_i^S \ \rightarrow\ v_i^{S_1^*(S)} = \frac{v(S_1^*(S))}{|S_1^*(S)|}, \text{ for all } i \in S_1^*(S), \tag{9}$$

$$\vdots$$

$$v_i^S \ \rightarrow\ v_i^{S_{K^S}^*(S)} = \frac{v(S_{K^S}^*(S))}{|S_{K^S}^*(S)|}, \text{ for all } i \in S_{K^S}^*(S),$$

as δ goes to 1. From (i) of Lemma 2, we have,

for every $i \in S_1^*(S)$,

$$v(S_1^*(S)) - \delta \sum_{k\in S_1^*(S),k\neq i} v_k^S \geq v(T) - \delta \sum_{k\in T,k\neq i} v_k^S \quad \text{for any } T \subset S \text{ with } i \in T, \tag{10}$$

$$\vdots$$

for every $i \in S_{K^S}^*(S)$,

$$v(S_{K^S}^*(S)) - \delta \sum_{k\in S_{K^S}^*(S),k\neq i} v_k^S \geq v(T) - \delta \sum_{k\in T,k\neq i} v_k^S \quad \text{for any } T \subset S \text{ with } i \in T.$$

Taking into account for (9), we have (1) of Theorem 2 as δ goes to 1 in (10).

(if): Suppose that (1) holds. From (ii) of Lemma 2 the expected payoff vector $(v_i^S)_{i\in S}$ satisfies the equations system (7). By the equations system (7) we can easily see that each δv_i^S is monotone increasing with δ and converges to $v(S_\ell^*(S))/|S_\ell^*(S)|$, $i \in S_\ell^*(S)$, $\ell = 1, \ldots, K^S$ as δ goes to 1. Moreover, δv_i^S

is continuous in δ. Therefore, for any δ sufficiently close to 1, we have the following inequalities;

$$v(S_1^*(S)) - \delta \sum_{k \in S_1^*(S), k \neq i} v_k^S \geq \max_{T \in \mathcal{S}_i} \left\{ v(T) - \delta \sum_{k \in T, k \neq i} v_k^S \right\}, \text{ for all } i \in S_1^*(S), \tag{11}$$

$$\vdots$$

$$v(S_{K^S}^*(S)) - \delta \sum_{k \in S_{K^S}^*(S), k \neq i} v_k^S \geq \max_{T \in \mathcal{S}_i} \left\{ v(T) - \delta \sum_{k \in T, k \neq i} v_k^S \right\}, \text{ for all } i \in S_{K^S}^*(S).$$

Let define the strategy combination σ^* of $\Gamma^N(\delta)$ such that, in every subgame $\Gamma^S(\delta)$, every player $i \in S_\ell^*(S)$ proposes the coalition $S_\ell^*(S)$ and the payoff vector y^i such that $y_i^i = v(S_\ell^*(S)) - \sum_{k \in S_\ell^*(S), k \neq i} y_k^i$ and $y_j^i = \delta v_j^S$ for $j \in S_\ell^*(S)$, and accepts any proposal (T, y^T) if and only if $y_i^T \geq \delta v_i^S$. From the above inequalities (11) and Lemma 2, σ^* becomes a SSPE of $\Gamma^N(\delta)$. Thus, as $\delta \to 1$, we have a pure strategy and limit subgame coalitional efficient SSPE of Γ^N with the expected payoff vector (2). □

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